Sequential Vote Buying

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Abstract

A leader wants to enact a general-interest policy but needs the support of q members of a committee who oppose the policy with heterogenous intensities. The leader sequentially approaches the committee members: in each period, she chooses which member to approach and what offer to make in exchange for his vote. We analyze two variants, depending on the nature of the offer. In the *transfer-promise* model, the leader pays the accepted offers only if she puts the policy to a vote; in the *up-front-payment* model, she pays the accepted offers immediately even if she does not put the policy to a vote eventually. In the transfer-promise model, the policy passes in equilibrium if and only if the leader's gain is higher than the sum of the losses of the q members who are least opposed; whenever the policy passes in equilibrium, the leader makes offers close to zero to the set of members who are least opposed to the policy, and the optimal sequence may require her to first approach the most-opposed member among the set. In the up-front-payment model, however, the leader does not necessarily buy the votes of the least-opposed members. The equilibrium now features two phases: in the first phase, each approached member is *indispensable* and thus compensated fully for his vote; in the second phase, each approached member in *dispensable* and thus offered a payment close to zero. Even though the leader may pay a significant amount for a vote, she is better off with the instrument of up-front payments because it is a commitment device that allows her to pass policy that she would not be able to with transfer promises. We also discuss extensions that allow simultaneous offers and bargaining with coalitions.

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1 Introduction

It is common practice for political leaders to use targeted district benefits, also known as pork, to gain support of specific voting members or voting blocks on general-interest legislation and public policy. As a well-known example, President Lyndon B. Johnson secured the critical support of Charles Halleck, Republican Congressman of Indiana, in passing the Civil Rights Act by offering him a NASA research facility at Purdue University.[1] For a comprehensive discussion of many other examples and empirical analyses of how pork barrel projects have been used to build majority coalitions in the U.S. Congress to enact public policies, see the book by Evans [2004].

We refer to this practice as "vote buying" in this paper. We introduce a simple game-theoretical model to capture certain essential elements of the sequential bargaining process which involves multiple members who collectively decide on the outcome of a policy of general interest. In our model, a leader, who would like to enact a new policy and needs q votes for it to pass, approaches members with varying degrees of opposition to the policy sequentially, making an offer to each member in exchange for his vote. If a member accepts an offer, then he commits to voting yes when the policy is up for a vote. Since the members are heterogeneous in their preferences, which members' votes should the leader buy? In what sequence should the leader approach them? How much do the votes cost and under what conditions will the policy get successfully enacted?

We address these questions in two variants of our model that differ in terms of the nature of the leader's offer. In the *transfer-promise* model, the leader's offer is a promise to make a transfer if a vote on the policy is held in exchange for the member's vote. This applies, for example, when the final bill that the members vote on bundles the policy and the transfers together. In the *up-front-payment* model, the leader's offer is an up-front payment in exchange for the member's vote, which is not contingent on whether the leader decides to put the policy up for a vote. This is more appropriate when the leader's transfer offer is separate from the bill that involves the policy.

[1] See, for example, https://www.theatlantic.com/magazine/archive/2014/04/what-the-hells-the-presidency-for/358630/.

In the transfer-promise model, we find that the policy passes if and only if the leader's gain is higher than the sum of the losses of the q members who are least opposed to the policy. Strikingly, whenever the leader is successful in getting the policy passed, the payments that she makes to the approached members are close to zero in equilibrium. This result is reminiscent of Dal Bó [2007], but arises from a different reason. Specifically, unlike Dal Bó [2007], in which a critical assumption is that a contract offered to a member is contingent on the profile of other members' votes (in particular, the amount of transfer explicitly depends on whether the member's vote turns out to be pivotal), the offers in our model take a simpler form. The intuition behind nearly zero equilibrium payments in our model is as follows: whenever a member is approached in equilibrium, he is "dispensable" in the sense that if he rejects the offer, the policy still passes in equilibrium; and this dispensability implies that his rejection only delays the passing of the policy and therefore he would accept an offer close to zero when he is sufficiently patient. It is interesting to note that the amount that the leader pays has little to do with the condition for the policy to pass in equilibrium: even though the leader needs to pay only minimal amount, in order for the policy to pass, her gain still has to surpass the sum of the losses of a certain subset of members.

Whose votes should the leader try to buy? One immediate response might be that the leader should buy the votes of those least opposed to the policy. This is largely borne out in the transfer-promise model. Specifically, without loss of generality, let us order the members in terms of their intensity of opposition to the policy so that higher index indicates higher intensity. If the qth member is dispensable at the beginning of the game, then an optimal sequence for the leader is to approach the q members who are least opposed to the policy in descending order: this sequence guarantees that every member is dispensable along the sequence, implying that the leader cannot buy q votes with a lower total offer since these are the q members who have the lowest losses. If the qth member is indispensable at the beginning of the game, however, this is no longer feasible. In this case, we show that the $(q+1)$th member is dispensable at the beginning of the game and the leader should approach the $(q+1)$th member first and then approach those $(q-1)$

members who are least opposed. Again, by doing this, she ensures that every member is dispensable along the sequence and therefore will accept offer close to 0.

In the up-front payments model, however, it is no longer the case that the leader buys the votes of those members who are least opposed to the policy. The structure of equilibrium differs from that in the transfer-promise model now that an accepted offer is sunk cost for the leader. We show that the equilibrium now features two phases: in the first phase, each approached member is "indispensable" in the sense that if he rejects the offer, the policy does not pass in equilibrium and therefore needs to be compensated fully for his losses when the leader buys his vote (referred to as the temptation phase); but as soon as some member becomes dispensable, the equilibrium enters the second phase in which any approached member is dispensable until the leader secures enough votes for the passage of the policy. In this phase, the members are offered payments close to zero, just like in the transfer-promise model (referred to as the exploitation phase).

To answer the question what members' votes the leader should buy, note that since the payments made in the exploitation phase are negligible as players become sufficiently patient, the leader's goal is to minimize the total payment she makes in the temptation phase. Our equilibrium characterization illustrates the basic tradeoff that the leader faces: the temptation phase is longer when members included in that phase are less opposed to the policy and shorter when members approached in that phase are more opposed to the policy. This is because a member is more likely to be indispensable if the other members are more strongly opposed to the policy. This highlights the fact that the endogenous sequencing creates endogenous cost of buying a vote.

Even though the leader may end up paying certain members a significant amount in exchange for their votes in the up-front payments model whereas she always pays a negligible amount in the transfer-promise model, the leader is better off if she can offer up-front payments instead of transfer promises. This is related to the result that in both models, when the leader has a higher gain from the policy (a higher willingness to pay), it is easier for the policy to pass and moreover, the cost of buying votes is lower in equilibrium. Since up-front payment is sunk cost, the leader's willingness to pay does not diminish as she secures more votes and this allows the leader to get the policy passed when it would not otherwise with transfer promises. So up-front payment can be deployed as a commitment device on the part of the leader.

Related literature. Our paper is related to three strands of literature. The first is the literature on *vote buying*. Most of this literature analyzes a model with two vote buyers who move sequentially. The vote buyers either move once [Groseclose and Snyder 1996; Banks 2000; Le Breton and Zaporozhets 2010; Le Breton, Sudholter, and Zaporozhets 2012] or repeatedly [Dekel, Jackson, and Wolinsky 2008, 2009; Morgan and Vardy 2011, 2012]. Unlike in our paper, a vote buyer's move consists of simultaneous offers to all vote sellers and vote sellers make their selling decisions simultaneously immediately before the game ends.[2] These papers either predict that 'near-median' voters sell their votes or do not make a prediction regarding the identity of the players who sell their votes. Different from the rest of the vote-buying literature, the strategic interaction between vote sellers is important in Neeman [1999] and Dal Bó [2007], but they do not address the question of endogenous sequencing.

The second strand of literature our paper is related to is *multi-agent contracting with externalities*. A typical application in this literature considers an incumbent firm trying to sign exclusionary contracts with buyers in order to prevent entry by its competitors. Unlike our paper, most of this literature either does not consider sequential nature of contracting [Bernheim and Whinston 1998; Segal 1999, 2003; Bernstein and Winter 2012] or assumes homogeneity of the members [Rasmusen, Ramseyer, and Wiley 1991; Rasmusen and Ramseyer 1994; Segal and Whinston 2000; Fumagalli and Motta 2006; Genicot and Ray 2006; Chen and Shaffer 2014; Iaryczower and Oliveros 2017, 2019].[3] Two exceptions are Möller [2007] and Galasso [2008], but they restrict attention of sequential contracting to two members.

The third strand of related literature is *bargaining with endogenous sequencing*. A

[2]Spenkuch, Montagnes, and Magleby [2018] consider an extension of Dekel et al. [2009] in which the vote buyers approach vote sellers sequentially in a predetermined order and show that the main predictions of Dekel et al. [2009] continue to hold.

[3]See Whinston [2006] and Rey and Tirole [2007] for surveys of this literature.

typical model in this literature studies a situation in which one player bargains with other n players sequentially. For tractability, papers in this literature often either work with a set of exogenously given sequences [Horn and Wolinsky 1988; Stole and Zwiebel 1996] or restrict attention to the case of $n = 2$ [Marshall and Merlo 2004; Menezes and Pitchford 2004; Noe and Wang 2004; Marx and Shaffer 2007; Bedre-Defolie 2012; Krasteva and Yildirim 2012a,b, 2019; Göller and Hewer 2015]. Cai [2000, 2003], Li [2010] and Xiao [2018] allow for the bargaining sequence to arise endogenously but, unlike our paper, focus on the case in which an agreement with all of the n players need to be reached (unanimity).

2 Model

A leader wants to pass a new policy. She sequentially approaches the members of a committee. With each approached member, she tries to reach a bilateral agreement, offering a transfer in return for the member's support. All actions are observable.

Formally, the game is played by the leader and a set of committee members $N = \{1, \ldots, n\}$, where $n \geq 1$. Passing the policy requires $q \in \{1, \ldots, n\}$ votes from the committee members (other than the leader). Each player's payoff from the status quo is normalized to be 0. The payoff from the new policy is $y > 0$ for the leader and $-x_i$ for each member $i \in N$. We assume that $x_i > 0$ for each $i \in N$ and index the committee members such that $x_i \leq x_{i+1}$, so a member with a higher index is more strongly opposed to the policy.[4]

The leader approaches the committee members in consecutive periods. Suppose that at the beginning of a period, the set of un-approached members is U and the number of approached members who have accepted the offers is n_a. The leader can choose to *approach* a member in U, or *initiate a vote* or *stop*. If the leader decides to approach a member $i \in U$, then she offers him a non-negative transfer in exchange for his vote. Member i either accepts the offer, thus giving the leader control of his vote, or rejects the

[4]Assuming that $x_i > 0$ is without loss of generality since any member with $x_i \leq 0$ prefers the new policy to the status quo and can thus be ignored in the analysis.

offer, and the game proceeds to the next period. If the leader decides to initiate a vote, the policy passes if $n_a \geq q$ and the status quo is maintained if $n_a < q$, and the game ends. If the leader decides to stop, the policy does not pass and the game ends. We consider two variants of this game which differ in terms of the nature of the leader's offer.

(1) Transfer promises: the leader's offer is a promise to make a transfer if and when a vote on the policy is held in exchange for the member's vote.[5]

(2) Up-front payments: the leader's offer is an up-front payment in exchange for the member's vote.

In the transfer-promise game, the leader does not make a payment if she chooses not to put the policy to a vote in the end, and therefore the transfer promises are *non-sunk cost*. In the up-front-payment game, the transfer is made irrespective of whether or not a vote on the policy is held. In this sense, the payment is *sunk cost* to the leader.

The game ends either when the leader stops or a vote is held on the policy. To describes the players' payoffs at the terminal nodes, let N_a be the set of members who accepted the leader's offers and for each $i \in N_a$, let τ_i be the period in which the offer t_i was accepted. If a vote is held, let τ denote the period. We assume that the players have a common discount factor $\delta \in (0,1)$.

First consider the transfer-promise game. If no vote is held by the end of the game, each player receives a payoff of 0. If a vote is held and the policy passes, the leader receives a payoff of $\delta^{\tau-1}(y - \sum_{i \in N_a} t_i)$, and member i receives a payoff of $\delta^{\tau-1}(t_i - x_i)$ if $i \in N_a$ and $\delta^{\tau-1}(-x_i)$ if $i \notin N_a$. If a vote is held and the policy does not pass, the leader receives a payoff of $\delta^{\tau-1}(-\sum_{i \in N_a} t_i)$, and member i receives a payoff of $\delta^{\tau-1}(t_i)$ if $i \in N_a$ and 0 if $i \notin N_a$.

In the up-front-payments game, if no vote is held or is held but the policy does not pass, the leader receives a payoff of $-\sum_{i \in N_a} \delta^{\tau_i - 1} t_i$, and member i receives a payoff of $\delta^{\tau_i - 1} t_i$ if $i \in N_a$ and 0 if $i \notin N_a$. If a vote is held and the policy passes, the leader receives

[5] An alternative assumption is for the leader's offer to be a promise to make a transfer if and when the policy passes in exchange for the member's vote. The equilibria under this assumption are outcome equivalent. Specifically, under this alternative assumption there could be an equilibrium in which the leader initiates a vote anticipating that the policy would not pass. This is outcome equivalent to stopping under our assumption, which we choose for analytical simplicity.

a payoff of $\delta^{\tau-1}y - \sum_{i \in N_a} \delta^{\tau_i - 1} t_i$, and member i receives a payoff of $-\delta^{\tau-1}x_i + \delta^{\tau_i-1}t_i$ if $i \in N_a$ and $-\delta^{\tau-1}x_i$ if $i \notin N_a$.

In the transfer-promise game, the transfers are paid when a vote on the policy is held and hence the payoffs from the policy and from the transfers are discounted by the same factor. In contrast, in the up-front-payment game, the transfers are paid immediately upon the acceptance of the offers while the passage of the policy happens only at the end of the game and hence the payoffs from the policy and from the transfers are discounted by different factors.[6]

All histories are public and record identity of the approached members, the transfers offered and members' acceptance decisions. Strategies are maps from histories to available actions. The solution concept we use is subgame perfect equilibrium (in which a member who is indifferent between accepting and rejecting accepts).[7] For the rest of the paper, we simply use the term equilibrium to refer to this solution concept.

The model we study assumes that offers and acceptance decisions are public, that the leader has all the bargaining power,[8] and cannot re-approach members.[9] Relaxing any of these assumptions, while certainly interesting, would significantly complicate the analysis and obscure the main strategic forces we study below. In Section 5 we study two extensions of our model relaxing the assumption that the leader approaches one member in each period and the assumption that each member controls one vote.

We introduce the notion of states to facilitate the analysis. Recall that a history at the

[6]With up-front payments, whether the transfers are paid immediately or at the end of the game is a modelling choice. Working with immediate payments simplifies the analysis because it implies that past leader's accepted offers do not influence leader's optimal action at a history.

[7]This refinement restricts the behavior of an indifferent member who is not approached on the equilibrium path. It is not needed in some well-known bargaining games, e.g., the ultimatum game, because the responder is always approached on the equilibrium path. The refinement does not change the set of equilibria that are observationally equivalent because for any subgame perfect equilibrium one can construct an outcome equivalent subgame perfect equilibrium in which indifferent members accept.

[8]Iaryczower and Oliveros [2019] study how the allocation of bargaining power affects contractual outcomes between a principal and multiple agents.

[9]Segal and Whinston [2000] and Genicot and Ray [2006] allow for re-approaching in models of contracting with externalities. Re-approaching has offsetting effect on the leader's ability to contract with the members; it makes members more demanding because the leader is no longer committed not to re-approach them, and it makes members less demanding because it intensifies competition among the members. In Segal and Whinston [2000] the latter effect dominates and re-approaching benefits the leader, while it has opposite effect in Genicot and Ray [2006].

beginning of a period records the set of members who have been previously approached, the transfers offered to them, and the members' acceptance decisions. For the model with transfer promises, a state is (S, r, t), where $S \subseteq N$, $S \neq \varnothing$, $r \in \{1, \ldots, q\}$ and $t \geq 0$, corresponds to a set of histories such that the set of members who have been previously approached is $N \setminus S$, the number of members who have sold their votes to the leader is $q - r$ and the sum of the promised transfers to these members is t. That is, state (S, r, t) corresponds to histories in which the set of un-approached members is S, the leader still needs support from r members in order for the policy to pass and the leader has already promised to pay a total of t to the members who have accepted the offers. For the model with up-front payments, we do not include the transfers that are already accepted and paid in a state since they do not affect the equilibria in the subgames. Hence, we denote a state by (S, r) in the up-front-payment model. Let $\mathcal{S} = \{(S, r, t) \in 2^N \times \mathbb{Z} \times \mathbb{R}_+ | S \neq \varnothing \wedge 1 \leq r \leq |S|\}$ be the set of all states (dropping the $\mathbb{R}_+$ dimension for the sunk cost model).[10] Note that given any state in $\mathcal{S}$ and any two histories inducing that state, the subgames following the two histories are identical and hence the two subgames have the same set of equilibria. Let $\Gamma(S, r, t)$ denote a subgame starting with state (S, r, t) in the transfer-promise model and $\Gamma(S, r)$ denote a subgame starting with state (S, r) in the up-front-payment model. Then the entire game is $\Gamma(N, q, 0)$ in the transfer-promise game and $\Gamma(N, q)$ in the up-front-payment game.

3 Transfer promises

We begin by studying the model in which the leader offers a transfer promise in exchange for a member's vote. We first establish conditions under which the policy passes in equilibrium and then characterize the optimal sequence in which the leader approaches the members and how much transfer promises she offers them in equilibrium.

[10] There are states not in $\mathcal{S}$ that could arise in the extensive form: for example, there are subgames in which $r > |S|$, but since they are not interesting to analyze, we exclude them from $\mathcal{S}$.

3.1 When does the policy pass?

Consider a subgame $\Gamma(S, r, t)$. Proposition 1 below says that whether the policy passes in equilibrium depends on how the leader's gain (net of the transfer promises already accepted) compares with the sum of the losses of the r members in S who are least opposed to the policy. Applying this result to the whole game immediately implies that whether the policy passes in equilibrium depends on whether the leader's gain from the policy is strictly higher than the sum of the losses of the q members who are least opposed to the policy.

Given $S \subseteq N$ and $0 \leq r \leq |S|$, let $S^r \subseteq S$ denote the set of the r members in S who have the lowest losses from the policy. Let $S^r = \varnothing$ if $r = 0$.

Proposition 1. *Suppose the leader offers transfer promises. Consider a subgame $\Gamma(S, r, t)$ where $r \leq |S|$. In any equilibrium, (a) if $y - t > \sum_{j \in S^r} x_j$, the policy passes, and (b) if $y - t < \sum_{j \in S^r} x_j$, the policy does not pass.*

To gain some intuition for part (a), note that a member i would accept an offer greater than his loss.[11] Hence, when the leader needs r votes, if her gain from the policy (net of the transfer promises that have already been offered and accepted) is larger than the sum of r members' losses, she can guarantee a strictly positive payoff by approaching these r members at the end (so that each understands that he is pivotal) and making each an offer that just compensates for the loss. Since the leader's payoff is only 0 when the policy does not pass, she is better off if she buys these members' votes and therefore the policy passes in any equilibrium.

For part (b), it is straightforward to see that if the cardinality of S equals r (corresponding to unanimity), then the leader needs the vote of every member in S, who will accept an offer only if it at least compensates for the member's loss. Hence, if the leader's gain from the policy (net of the transfer promises already accepted) is lower than the sum of the members' losses, then she can only receive a negative payoff by getting the policy passed, whereas she receives a payoff of 0 if the policy does not pass. Given that

[11] If we restrict attention to undominated strategies, then clearly a member i would accept an offer greater than his loss, but our result still holds even if weakly dominated strategies are allowed.

the leader is better off if the policy does not pass and she can choose to stop, the policy does not pass in any equilibrium. If cardinality of S is equal to $r+1$, then the member approached in the first period will accept an offer only if it at least compensates for his loss since the member foresees that without his support, the policy will not pass in the continuation game. Since every member can reason like this and thus demands an offer that makes him at least even, again the leader cannot buy enough votes without offering transfer promises that exceed her gain from the policy. Induction shows that the policy does not pass in any equilibrium.

3.2 Equilibrium sequencing with transfer promises

We have established in Proposition 1 that if $y > \sum_{i=1}^{q} x_i$, then the policy passes in any equilibrium. In what sequence should the leader approach the members and what transfer promises does she offer them in equilibrium? These are the questions we address in this subsection. The answer is immediate under unanimity ($q = n$): the leader offers each member i a transfer promise equal to x_i and the sequence of approaching does not matter. In what follows, we consider $q < n$. It is useful to introduce the notion of "(in)dispensability."

Definition 1. *In the transfer-promise game, consider state (S, r, t) where $r < |S|$. (a) We say that member $i \in S$ is indispensable in (S, r, t) if $\sum_{j \in S^r} x_j < y - t < \sum_{j \in S^r_{-i}} x_j$. (b) We say that member $i \in S$ is dispensable in (S, r, t) if $y - t > \sum_{j \in S^r_{-i}} x_j$.*

Intuitively, a member is indispensable in a state if the policy does not pass in equilibrium without the leader securing the member's vote in that state whereas a member is dispensable in a state if the policy still passes in equilibrium even without the leader securing the member's vote in that state. A member's strategic position is stronger when he is indispensable than when he is dispensable. When a member is indispensable, he accepts the leader's offer only if it at least compensates for his loss since by rejecting the offer, the policy will fail to pass. When a member is dispensable, however, he anticipates that the policy still passes even if he rejects the offer. Since his rejection only delays the passage of the policy, he is willing to accept an offer that just compensates him for a sooner passage ($(1-\delta)x_i$ to member i). Note that when δ is sufficiently high, a dispensable member is willing to accept an offer close to 0, an offer lower than any offer that the leader needs to make in order to secure an indispensable member's vote, no matter what that member's loss is.

Fix a transfer-promise game and suppose that the policy passes in equilibrium. Let A^d denote the set of members who are dispensable at the beginning of the game. Note that since the policy passes in equilibrium, that is, $y > \sum_{i=1}^{q} x_i$, any member in $\{q+1, q+2, ..., n\}$ is in A^d. As Proposition 2 below shows, what members' votes the leader buys depends on whether the qth member is in A^d.

If the qth member is in A^d, then it is optimal for the leader to approach the members who have the lowest q losses. She should start by approaching a member in $A^d \cap \{1, ..., q\}$ since he is dispensable; after buying this member's vote, every remaining member i in $\{1, ..., q\}$ becomes dispensable and therefore will accept an offer $(1-\delta)x_i$, and the leader can approach them in an arbitrary sequence. Since the members being approached in this sequence have the lowest losses among all members, the leader cannot improve her payoff by approaching others.

If the qth member is not in A^d, then any member with a lower loss is also not in A^d. Hence, if the leader starts by approaching any of them, she has to make an offer equal to the loss. However, since member $(q+1)$ is in A^d, she only needs to offer $(1-\delta)x_{q+1}$. Moreover, after securing this member's vote with a transfer promise close to 0, any member in $\{1, 2, ..., q\}$ becomes dispensable in the continuation game, and therefore it is optimal for the leader to approach the members who have the lowest $(q-1)$ losses in the continuation game in an arbitrary sequence. The following proposition formalizes the results.

Proposition 2. *In the transfer-promise game, suppose $n > q$ and $y > \sum_{i=1}^{q} x_i$.*
(a) If $y > \sum_{i=1}^{q-1} x_i + x_{q+1}$, that is, member q is dispensable at the beginning of the game, then in any equilibrium, the leader starts by approaching a member in $A^d \cap \{1, ..., q\}$ and then approaches the remaining members in $\{1, ..., q\}$ in arbitrary order; when she approaches member i, she offers $t_i = (1-\delta)x_i$ and it is accepted.

(b) If $y < \sum_{i=1}^{q-1} x_i + x_{q+1}$, that is, member q is indispensable at the beginning of the game, then there exists $\bar{\delta} < 1$ such that for $\delta > \bar{\delta}$, in any equilibrium, the leader starts by approaching member $q+1$ and then approaches members in $\{1, ..., q-1\}$ in arbitrary order; when she approaches member i, she offers $t_i = (1-\delta)x_i$ and it is accepted.

4 Up-front payments

We now turn to the model in which the leader offers an up-front payment in exchange for a member's vote. Note that under unanimity, since every member i has the right to veto, he accepts an offer if and only if it compensates for his loss x_i, appropriately discounted. Specifically, in any state (S, r) such that $|S| = r$, if member i is approached in equilibrium, he accepts the offer t_i if and only if $t_i \geq \delta^r x_i$. (Since the policy passes after the leader buys the votes of the remaining r members but the payment is up front, the member is willing to accept any offer greater than $\delta^r x_i$.) It follows that the leader's payoff is $\delta^r(y - \sum_{i \in S} x_i)$ by getting the policy pass, and therefore the policy passes in equilibrium if this payoff is positive, that is, $y > \sum_{i \in S} x_i$. Another special case is when $r = 1$. Since the leader needs only one vote for the policy to pass, once the leader buys one member's vote, she does not approach any more members and initiates voting immediately. Whether the offer is an up-front payment or a transfer promise does not matter for the incentives, implying that the condition for the policy to pass in equilibrium is $y > x_1$. In contrast, when the leader needs more than one vote for the policy to pass, whether the offer she makes is up-front payment or transfer promises has important implications, which we illustrate by the following example.

Example 1. *Suppose $n = 3$ and $q = 2$. We first show that when $y > x_1 + x_2$, then the policy passes in equilibrium with offers close to 0 when the players are patient, similar to what happens in transfer-promise game. We then show that if $x_2 < y < x_1 + x_2$, then the policy still passes in equilibrium in the up-front-payment game, even though it does not in the transfer-promise game, but in this case, not all offers are close to 0. We finally show that if $y < x_2$, then the policy does not pass even in the up-front-payment game.*

First consider $y > x_1 + x_2$. If member 3 is approached first, then he is willing to accept any offer greater than $x_3\delta^2(1-\delta)$ because even if he rejects the offer, the policy will still pass in the continuation game and therefore his rejection only delays the passing of the policy by one period. Hence, he accepts any offer t such that $t - \delta^2 x_3 > \delta^3 x_3$. After member 3's vote is bought, member 1 is willing to accept any offer greater than $x_1\delta(1-\delta)$ because his rejection only delays the passing of the policy by one period. Note that both offers are close to 0 for patient players – we refer to them as "exploitation" offers.[12]

Now consider $x_2 < y < x_1 + x_2$. If member i is approached first, he is willing to accept an offer if and only if $t_i \geq \delta^2 x_i$. To see this, note that if member i rejects the offer, then the policy will fail to pass since the leader would need to buy each remaining member's vote, which is too costly given that $y < x_1 + x_2$. But after securing member 1's vote by offering him $t_1 = \delta^2 x_1$ (we call this an "temptation" offer), now the leader can buy member 3's vote by making him an exploitation offer $(1-\delta)x_3$. Since the leader can buy enough votes at a cost lower than y, the policy passes in equilibrium.

Finally consider $y < x_2$. For the same reason as discussed above, the leader has to make a temptation offer to the member approached first. Since $y < x_2$, it is too costly for the leader to tempt member 2 or 3. Furthermore, even if the leader buys member 1's vote first, whoever the leader approaches next would still accept only a temptation offer, which would be too costly. Hence, the policy does not pass in equilibrium even in the up-front-payment game.

4.1 When does the policy pass?

The next proposition says that given a state (S, r), if the players are sufficiently patient, the policy passes if and only if y is above a threshold $W(S, r)$, defined recursively as follows. For any state (S, r), denote by $\max S$ the member in S with the highest loss and

[12] This is the optimal sequence if $y < x_1 + x_3$; if $y > x_1 + x_3$, then it is optimal to approach member 2 first, followed by member 1.

let $S' = S \setminus \{\max S\}$ ($\varnothing' = \varnothing$ by convention). Let

$$W(S, r) = \min_{T \in 2^S} \max \left\{ \sum_{j \in T} x_j, W((S \setminus T)', r - |T|) \right\}. \tag{1}$$

Proposition 3. *Suppose the leader offers up-front payments. Consider a subgame $\Gamma(S, r)$. For generic y, there exists $\bar{\delta} < 1$ such that for $\delta > \bar{\delta}$, in any equilibrium, the policy passes if $y > W(S, r)$ and the policy does not pass if $y < W(S, r)$.*

To understand why $W(S, r)$ is the threshold that determines whether the policy passes in equilibrium, it is useful to classify the members in terms of their bargaining positions given a state.

Definition 2. *A member $i \in S$ in state (S, r) is*

1. *dispensable if $y > W(S \setminus \{i\}, r)$,*
2. *indispensable if $y \in (W(S \setminus \{i\}, r - 1), W(S \setminus \{i\}, r))$.*

The definition of dispensability and indispensability here parallel those in the model of transfer promises. As implied by Proposition 3, a member is indispensable in a state if the policy does not pass in equilibrium without the leader securing the member's vote in that state whereas a member is dispensable in a state if the policy still passes in equilibrium even without the leader securing the member's vote in that state. As before, when a member is indispensable, he has a strong bargaining position and thus accepts the leader's offer if and only if it at least compensates for his loss (with the appropriate discounting), but when a member is dispensable, he has a weak bargaining position and is therefore willing to accept an offer that just compensates him for a sooner passage of the policy. We referred to these two distinct kinds of offers as temptation and exploitation offers, as formalized in the following definition.

Definition 3. *Fix a profile of strategies and consider the resulting sequence of approached members. Suppose member i is offered t_i in state (S, r). We say that member i is tempted if $t_i = \delta^r x_i$ and that member i is exploited if $t_i = \delta^r x_i(1 - \delta)$. We say a profile is in a* *temptation phase when the approached member is tempted and is in an exploitation phase when the approached member is exploited.*

Lemma 1. *Suppose the leader offers up-front payments. For generic y, there exists $\bar{\delta} < 1$ such that for $\delta > \bar{\delta}$, in any equilibrium, if member i is approached in a state in which he is indispensable, then he is tempted, and if member i is approached in a state in which he is dispensable, then he is exploited.*

Lemma 2. *Given a state (S, r), if there exists a member $i \in S$ who is dispensable, then (i) member $\max S$ is dispensable in (S, r); (ii) any member in $S \setminus \{i\}$ is dispensable in state $(S \setminus \{i\}, r - 1)$, which implies that starting in state (S, r), there exists a sequence of r members along which each member is dispensable.*

Since the payment made to a dispensable member goes to 0 as the discount factor goes to 1 by Lemma 1, once an exploitation phase starts in equilibrium, it remains in that phase until the leader buys all the votes she needs. Moreover, since the payment made to an indispensable member equals his loss (in the limit as the discount factor goes to 1), the total payment that the leader makes is the sum of the losses of the members approached in the temptation phase. Hence, if there exists a set of members such that the following two conditions hold: (i) the sum of their losses is below y, and (ii) after the leader buys their votes, at least one remaining member is dispensable, then the policy will pass in equilibrium. Conditions (i) and (ii) are reflected in the definition of $W(S, r)$.

4.2 Equilibrium sequencing with up-front payments

To see which members are tempted in equilibrium, consider the following problem for any state (S, r):

$$\Pi(S, r, y) = \min_{T \in 2^S} \sum_{j \in T} x_j \text{ s.t. } y > W((S \setminus T)', r - |T|). \tag{2}$$

The constraint ensures that after the leader buys the votes of members in the set T, there exists one member in the remaining set who is dispensable. As discussed above, the leader's payments to members who are dispensable are close to 0 and therefore she is only concerned about her payments to members in T, which equals the sum of their losses. Hence, the least costly way for her to buy enough votes to get the policy passed involves tempting members in T that solves (2). It also follows that the payment that the leader makes in equilibrium is close the value of the problem, that is, equals $\Pi(S, r, y)$.

We summarize these characterizations of the equilibrium in Proposition 4 below.

Proposition 4. *Suppose the leader offers up-front payments and* $y > W(N, q)$. *For generic* y*, there exists* $\bar{\delta} < 1$ *such that for* $\delta > \bar{\delta}$ *following results hold.*
(a) In any equilibrium, the leader approaches q *members and each accepts her offer.*
(b) Any equilibrium consists of two phases (with one possibly empty), a temptation phase followed by an exploitation phase.
(c) In any equilibrium, the set of members included in the temptation phase solves the optimization problem (2).
(d) Let T *be the members included in the temptation phase and* E *be the members included in the exploitation phase. For any order of the members in which the members in* T *are before the members in* E *and in which the first member in* E *is dispensable in* $(N \setminus T, r - |T|)$*, there exists an equilibrium in which the members are approached in that order.*
(e) The leader's equilibrium payoff is constant across equilibria and its limit is $y - \Pi(N, q, y) > 0$ *as* $\delta \to 1$.

When $y > W(N, q)$ the policy passes in any equilibrium by Proposition 3, and Proposition 4 shows that in any such equilibrium the leader approaches exactly q members with offers that are accepted, that the approached members are first tempted and then exploited, that the set of tempted members solves (2), and that any equilibrium multiplicity does not affect the leader's payoff. Multiple equilibria may exist but affect only the members' payoffs. This reflect two main sources of equilibrium multiplicity. First, when two members have identical loss, the leader might be indifferent as to which one to approach. Second, the leader is indifferent between all possible orders within the temptation and the exploitation phase, provided the latter starts with a dispensable member.

Propositions 3 and 4 provide characterization of all equilibria with up-front payments by linking properties of the equilibria to the W and Π functions. However, it is not possible to derive closed-form expressions for these functions except in special cases.[13]

Proposition 5. *Consider state (N, q) and suppose that $y > W(N, q)$:*

1. *(one vote needed) $W(N, q) = x_1$ and $\Pi(N, q, y) = 0$ if $q = 1$ and $n \geq 2$,*
2. *(unanimity) $W(N, q) = \sum_{j \in N} x_j$ and $\Pi(N, q, y) = \sum_{j \in N} x_j$ if $q = n$,*
3. *(simple majority and less) $W(N, q) = x_q$ if $q \leq \frac{n+1}{2}$ and $\Pi(N, q, y) = 0$ if $q < \frac{n+1}{2}$,*
4. *(homogeneous losses) $W(N, q) = \lceil \frac{q}{n-q+1} \rceil x = \lfloor \frac{n}{n-q+1} \rfloor x$ and $\Pi(N, q, y) = tx$, where t is the smallest non-negative integer such that $y > \lceil \frac{q-t}{n-q} \rceil$, if $x_i = x$ for all $i \in N$,*
5. *(i) $W(N, q)$ depends only on the losses of members 1 through q, that is, $W(N, q) = W(\hat{N}, q)$ if $|N| = |\hat{N}|$ and $x_i = \hat{x}_i$ for all $i \in \{1, \ldots, q\}$, and (ii) the equilibrium temptation phase includes at most q members and includes only members with q lowest losses, that is, any T that solves (2) satisfies $|T| \leq q$ and $i \notin T$ if $x_i > x_q$, and*
6. *$W(N, q) \leq \sum_{i=1}^{q} x_i$ and $\Pi(N, q, y) = 0$ if $y > \sum_{i=1}^{q} x_i$ and $q < n$.*

The proposition shows that when the leader needs one vote for the policy to pass, the policy passes in equilibrium depending on how y compares to x_1 and passes at no cost (in the limit). When the passage of the policy requires the votes of all members, the cost of passing the policy is the sum of the members' losses. For voting rules weakly below simple majority, the policy passes when $y > x_q$ and passes at no cost (in the limit) for voting rules strictly below simple majority. Parts 1 through 3 summarize these special cases. Part 4 derives W and Π when members have homogeneous losses and implies that both are non-increasing in n and non-decreasing in q. Part 5 applies generally and shows that the condition for the policy passing depends only on the losses of the members with q lowest losses. In addition, any equilibrium temptation phase includes at most q members and excludes members with losses strictly above the loss of member q. Part 6 connects the

[13] For any state (S, r) with $|S| - r = 1$, problem (3) defining W is a special case of the Partition problem and problem (2) defining Π is a special case of the Knapsack problem. Both of these problems are well known in computer science and combinatorial optimization and both are NP-hard.

up-front payment and the transfer promise models. Recall that with transfer promises, the policy passes in any equilibrium when y exceeds $\sum_{i=1}^{q} x_i$ at no cost (in the limit). With up-front payments, the policy passes not only when $y > \sum_{i=1}^{q} x_i$, in which case it passes at no cost (in the limit), but also when $y \in (W(N,q), \sum_{i=1}^{q} x_i)$.

4.3 Comparative statics

The following proposition shows how equilibrium outcomes vary with paramters.

Proposition 6. *Consider state (N, q) and any y:*

1. $W(N,q) \geq W(N,q-1)$ *and* $\Pi(N,q,y) \geq \Pi(N,q-1,y)$ *if* $q \geq 2$,
2. $W(N,q) \geq W(\hat{N},q)$ *and* $\Pi(N,q,y) \geq \Pi(\hat{N},q,y)$ *if* $|N| \leq |\hat{N}|$ *and* $x_i \geq \hat{x}_i$ *for all* $i \in \{1,\ldots,n\}$,
3. $W(N,q)$ *is independent of* y *and* $\Pi(N,q,y) \geq \Pi(N,q,y')$ *if* $y' > y$.
4. $W(N,q) \geq W(N \setminus \{i\}, q-1)$ *and* $\Pi(N,q,y) \geq \Pi(N \setminus \{i\}, q-1, y)$ *for any* $i \in N$ *if* $q \geq 2$,

The first three parts of the proposition show that the condition under which the policy passes in equilibrium is less stringent and that the (limiting) cost of the policy passing is lower either when the passage of the policy requires fewer votes (part 1), or when the committee is larger or is composed of weakly opposed members (part 2), or when the leader gains more from the policy (part 3). This implies that smaller committees and fewer required votes have offsetting effects on the passage of the policy and the cost of the passage. Nevertheless, the fourth part shows that the effect of the required votes dominates. That is, decreasing the size of the committee benefits the leader if combined with the same decrease in the number of votes required for the policy passing.

Proposition 6 shows that weakly-opposed members make it easier for the leader to buy votes. The proposition, however, is silent on the effect of offsetting changes in the members' opposition. That is, keeping the total opposition $\sum_{i \in N} x_i$ constant at c, is homogeneous or heterogeneous opposition more effective, in the sense of maximizing either

$W(N,q)$ or $\Pi(N,q,y)$ or both? From Proposition 5 part 2, any profile $(x_i)_{i\in N}$ yields the same $W(N,q)$ and $\Pi(N,q,y)$ under unanimity. Otherwise, Proposition 5 part 6 shows that profile $(x_i)_{i\in N}$ with $x_i = \varepsilon$ for all $i < n$ and $x_n = c - \varepsilon(n-1)$, for small enough $\varepsilon > 0$, yields $W(N,q)$ arbitrarily close to zero and $\Pi(N,q,y) = 0$. That is, committees composed of $n-1$ weakly opposed members and one strongly opposed member represent ineffective opposition. The next proposition shows which committees represent effective opposition in the sense of maximizing $W(N,q)$.

Proposition 7. *Consider state (N,q) and suppose that $q < n$ and $\sum_{i\in N} x_i = c$. Then $W(N,q) \leq \frac{c}{n-q+1}$. Moreover, if $x_i = \varepsilon$ for $i \in \{1,\dots,n-n_b\}$ and $x_i = \frac{c-\varepsilon(n-n_b)}{n_b}$ for $i \in \{n-n_b+1,\dots,n\}$, where $n_b = k(n-q+1)$ for some $k \in \{1,\dots,\lfloor\frac{n}{n-q+1}\rfloor\}$, then $\sum_{i\in N} x_i = c$ and there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0,\bar{\varepsilon})$, $W(N,q) = \frac{c-\varepsilon(n-n_b)}{n-q+1}$.*

Proposition 7 implies that committees composed of n_b equally strongly opposed members, where n_b is multiple of $n-q+1$, with the remaining members only weakly opposed, represent effective opposition in the sense of bringing $W(N,q)$ arbitrarily close to its upper bound, while keeping the sum of the members' losses constant. The profile constructed in the proposition does not reach the bound because the losses are constrained to be strictly positive. The proposition also implies that multiple profiles that bring $W(N,q)$ arbitrarily close to its upper bound may exist. For example, when $n = 5$ and $q = 4$, then $W(N,q) = \frac{c-3\varepsilon}{2}$ under profile $(\varepsilon,\varepsilon,\varepsilon,\frac{c-3\varepsilon}{2},\frac{c-3\varepsilon}{2})$ and $W(N,q) = \frac{c-\varepsilon}{2}$ under profile $(\varepsilon,\frac{c-\varepsilon}{4},\frac{c-\varepsilon}{4},\frac{c-\varepsilon}{4},\frac{c-\varepsilon}{4})$.

5 Extensions and discussions

In this section, we discuss several extensions of our main model. To start, it is useful to consider a model in which the leader must make offers to all members simultaneously, a benchmark that provides a contrast to our sequential vote-buying model.

5.1 Simultaneous offers

Consider the following extensive-form game. In the first period, the leader either offers a profile of transfer promises $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}_n^+$, or stops, or initiates a vote. If the leader stops or initiates a vote, then the game ends and the players receive their payoffs. If the leader offers $\mathbf{t}$, then the members sequentially, in some predetermined order, decide either to accept or reject the leader's offer and the game then proceeds to the second period in which the leader chooses either to initiate a vote or to stop. The game then ends and all players receive their payoffs. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be the profile of members' actions, where for each $i \in N$, $a_i = 0$ indicates rejection and $a_i = 1$ indicates acceptance.

If the leader stops in any of the periods or initiates a vote in the first period, the policy does not pass and all players receive zero payoff. If the leader initiates a vote in the second period, the leader's payoff is $y - \sum_{i \in N} a_i t_i$ and member i's payoff is $-x_i + a_i t_i$ if $\sum_{i \in N} a_i \geq q$ (the policy passes) and the leader's payoff is $-\sum_{i \in N} a_i t_i$ and member i's payoff is $a_i t_i$ if $\sum_{i \in N} a_i < q$ (the policy does not pass) if $\sum_{i \in N} a_i < q$.[14] Players discount payoffs by $\delta \in (0, 1]$ between periods.

We study pure strategy subgame perfect equilibria. We assume that member who is indifferent between accepting and rejecting and is pivotal, that is, would change whether the policy passes by changing her action, accepts. Below we call a pure strategy subgame perfect equilibrium with this acceptance rule simply an equilibrium.

Proposition 8. *Consider the simultaneous vote-buying game with transfer promises. If $\sum_{i=1}^{q} x_i > y$, then the policy does not pass in any equilibrium. If $\sum_{i=1}^{q} x_i < y$, then in any equilibrium, (a) the policy passes, (b) q members are offered strictly positive transfers, (c) if member i is offered a strictly positive transfer, then $t_i = x_i$ and (d) $\sum_{i \in N} t_i = \sum_{i=1}^{q} x_i$.*

As can be seen from Proposition 8, the condition for the policy passing in equilibrium is the same whether the leader make transfer promises simultaneously or sequentially: it depends on how her gain from the policy compares with the sum of the lowest q losses of

[14] Notice that the payoff of a member i who rejects is $-x_i$ and 0 when the policy passes and does not pass respectively, and the payoff of a member i who accepts is $-x_i + t_i$ and is in $\{0, t_i\}$ when the policy passes and does not pass respectively.

the members. Moreover, the set of members who receive positive transfers are (largely) the same whether they are approached simultaneously or sequentially: it is the set of members who are least opposed to the policy (with the caveat that under sequential vote-buying, sometimes the $q+1$th member needs to be bought). However, the *amount* of transfers they receive are drastically different: when approached sequentially, the members receive exploitation offers (offers close to 0), whereas when approached simultaneously, the members receive temptation offers (offers that compensate fully their losses). This is because when the leader has to make simultaneous offers to all members, she can no longer use other members as threat points. Without the divide-and-conquer mechanism at her disposal when she could approach the members sequentially, she now has to buy q votes by compensating the members fully for their losses.

Analogous results hold if the offers are up-front payments instead of transfer promises. The condition for the policy passing and the set of members who receive strictly positive transfers in equilibrium are identical. The only difference is that the temptation offers with up-front payments are δx_i instead of x_i. This is because the periods in which the transfers are received and the policy passes are different.[15]

We have analyzed an extensive form in which the members make acceptance/rejection decisions sequentially even though the offers are made simultaneously. One important reason for our choice of analyzing this extensive form instead of an alternative one in which the acceptance/rejection decisions are made simultaneously is the issue of multiple equilibria, which we discuss below. Consider the extensive form in which the leader makes simultaneous offers $\mathbf{t} \in \mathbb{R}_n^+$ and the members then simultaneously decide whether to accept or reject the offers. Note that if the leader makes an arbitrarily small positive offer $\varepsilon > 0$ to $q+1$ members, then it is an equilibrium for each member to accept the offer since no one is pivotal – that is, given that all other members who have received offers are going to accept, it is a strict best response to accept. This equilibrium relies on a kind

[15]When the offers are made simultaneously, the difference between sunk and non-sunk cost disappears and hence the conditions for policy passing and the set of members who receive positive transfers in equilibrium are identical. Details of the results for the case of up-front payment when the offers must be simultaneous are available upon request.

of mis-coordination on the part of the members; and since its existence is independent of the preferences of the players, this can lead to highly inefficient outcomes. On the other hand, it is also an equilibrium in which every member who has received an offer says no. There are various ways to deal with this multiple-equilibria problem. In Genicot and Ray [2006], for example, they impose a refinement to rule out members' mis-coordination. Instead of imposing a similar refinement, we analyze the extensive form in which the members acceptance/rejection decisions are made sequentially, which in effect eliminates their mis-coordination in equilibrium.

5.2 General timing: simultaneous offers allowed

In our main model of sequential vote buying, the leader makes one offer to one member in each period. We now enrich our model such that in each period, the leader can make simultaneous offers to a subset of members who have not been approached before, or initiate a vote, or stop. If the leader makes simultaneous offers to a subset of members, these members then sequentially decide, in some predetermined order, whether to accept or reject the leader's offers, and then the game proceeds to the next period. The assumptions on payoffs are the same as before.

We analyze the case when the offers are transfer promises in what follows. At the end of this subsection, we discuss what happens when the offers are up-front payments. As before, a state is (S, r, t), where S is the set of members who have not been approached by the leader, r is the number of votes still needed for the policy to pass and t is the sum of the promised transfers to the members who have already been approached by the leader and accepted the offers. If in the current period, $X \subseteq S$ members are approached and $\hat{X} \subseteq X$ accept the offers, whose sum is $\hat{t}$, then the state transitions to $(S \setminus \hat{X}, r - |\hat{X}|, t + \hat{t})$.

We first show that when simultaneous offers are allowed, the condition for the policy passing in equilibrium that we established in Proposition 1 remains the same.

Proposition 9. *Suppose the leader offers transfer promises with simultaneous offers allowed. Consider a subgame $\Gamma(S, r, t)$ where $r \leq |S|$. In any equilibrium, (a) if $y - t >$*

$\sum_{j \in S^r} x_j$*, the policy passes, and (b) if* $y - t < \sum_{j \in S^r} x_j$*, the policy does not pass.*

To determine the optimal sequence in which the leader should approach the members now that simultaneous offers are allowed, we adapt the notion of (in)dispensability of an individual member defined before to joint (in)dispensability of a set of members.

Definition 4. *In the transfer-promise game when simultaneous offers are allowed, consider state* (S, r, t) *where* $r < |S|$*. (a) A subset of members* $M \subseteq S$ *is jointly indispensable in* (S, r, t) *if* $|S \setminus M| < r$ *or if* $\sum_{j \in S^r} x_j < y - t < \sum_{j \in (S \setminus M)^r} x_j$*. (b) A subset of members* $M \subseteq S$ *is jointly dispensable in* (S, r, t) *if* $|S \setminus M| \geq r$ *and* $y - t > \sum_{j \in (S \setminus M)^r} x_j$*.*

Intuitively, a subset of members is jointly indispensable in a state if the policy does not pass in equilibrium without the leader securing all the members' votes in that state whereas a subset of members is jointly dispensable in a state if the policy still passes in equilibrium even without the leader securing all the members' votes in that state.

A subset of members' strategic position is stronger when it is indispensable than when it is dispensable. When a subset is indispensable, each member i accepts the leader's offer only if it at least compensates for his loss x_i since by rejecting the offer, the policy will fail to pass. When a subset is dispensable, however, each member anticipates that the policy still passes even if he rejects the offer.

Proposition 10. *In the transfer-promise game with simultaneous offers allowed, if the* q *lowest members are jointly dispensable at the beginning of the game, then in equilibrium the leader approaches them in the first period, making each of them an offer of* $(1 - \delta)x_i$*, and these offers are accepted.*

Proposition 10 highlights a significant difference between our paper and Genicot and Ray [2006]. In Genicot and Ray [2006], the exploitation phase must proceed sequentially, an implication of their assumptions on payoffs: specifically, that the agents' reservation payoffs are strictly increasing in the number of "free agents," that is, agents who have not accepted the principal's offers. In our model, in contrast, the exploitation phase may take place in just one shot, provided that the lowest q members are jointly dispensable, which happens if the leader's gain from the policy is sufficiently high and the number of votes needed is below majority. (The conditions are less stringent for simultaneous, not necessarily one-shot, approaching of members to arise in our model.)

5.3 Bargaining with coalitions

Suppose there are m coalitions with different voting weights. Let c_i be the number of votes that coalition i has, so $\sum_{i=1}^{m} c_i = n$. We also order the coalitions so that $c_i \leq c_{i+1}$. Assume that coalition i's loss from the policy is $x_i > 0$. (A special case is when the loss is proportional to the size of the coalition, that is, $x_i = x \cdot c_i$ where $x > 0$. In this case, without loss of generality, we normalize $x = 1$.)

Note that if $c_i > n - q$, then coalition i has veto power and clearly the leader has to pay it c_i in order for the policy to pass. In what follows, we assume that $c_m \leq n - q$, that is, no coalition has veto power. Let $M = \{1, ..., m\}$. In our previous analysis when the leader bargains with individual members, $S \subseteq N$ is a coalition of members; in this section, $S \subseteq M$ is a coalition of coalitions.

For any $S \subseteq M$, let $S^{\#}$ denote the total number of votes of the coalitions in S, that is, $S^{\#} = \sum_{i \in S} c_i$. Let $S^{\min}$ and $S^{\max}$ be the smallest and largest coalitions in S, respectively.

Let $\mathcal{K}(M, q)$ be the collection of subsets of coalitions such that the total voting weights of each subset of the coalitions is at least q and the total losses the coalitions in each subset is the smallest, that is,

$$\mathcal{K}(M, q) = \underset{K \subseteq M}{\arg\min} \sum_{i \in K} x_i$$

$$\text{subject to } \sum_{i \in K} c_i \geq q.$$

We call $K(M, q) \in \mathcal{K}(M, q)$ a minimum-loss winning coalition: since the total number of votes that the coalitions in $K(M, q)$ have is at least q, it is "winning"; and among all the winning coalition of coalitions, it has the lowest total loss from the policy. Let $X(M, q)$ be the value of the problem, that is, $X(M, q) = \sum_{i \in K(M,q)} x_i$ is the total loss of a minimum-loss winning coalition (MLWC for short). Note that if each coalition is an individual member, then $M = N$ and $X(M, q) = \sum_{i=1}^{q} x_i$, the sum of the lowest q losses of the members.

The following proposition generalizes the results we established for the transfer-promise game when the leader bargains with individual voters: (i) whether the policy passes in equilibrium depends on whether y is higher than the total loss of a MLWC, and (ii) when the policy passes, the leader makes payment close to 0 when the players are patient.

Proposition 11. *Suppose the leader makes transfer promises to coalitions. (a) If $y > X(M, q)$, then the policy passes in any equilibrium; if $y < X(M, q)$, then the policy does not pass in any equilibrium. (b) Moreover, when the policy passes in equilibrium, the payment that the leader makes in equilibrium goes to* 0 *as δ goes to* 1*.*

The intuition for why the comparison of the leader's gain y and loss of the MLWC $X(M, q)$ determines whether the policy passes in equilibrium is similar to that for Proposition 1. To prove part (b), we extend the definition of (in)dispensability to coalitions in the transfer-promise game as follows. As before, let (S, r, t) denote a state, where the set of un-approached coalitions is S, the leader still needs r votes in order for the policy to pass and the leader has already promised to pay a total of t to the coalitions who have accepted the offers.

Definition 5. *In the transfer-promise game, consider state (S, r, t) where $r < |S|$. (a) Coalition $i \in S$ is indispensable in (S, r, t) if $X(S, r) < y - t < X(S \setminus \{i\}, r)$. (b) Coalition $i \in S$ is dispensable in (S, r, t) if $y - t > X(S \setminus \{i\}, r)$.*

For the same reason as discussed in section 3.2, if coalition i is approached in a state in which it is dispensable, then it is willing to accept any offer greater than $(1 - \delta)x_i$, which goes to 0 as δ goes to 1. To establish part (b), we show that there exists a sequence of coalitions along which each coalition is dispensable when the leader approaches it and the sum of the weights of the coalitions is higher than q. Specifically, pick a MLWC $K(M, q)$. We first show that any coalition in $M \setminus K(M, q)$ is dispensable at the beginning of the game and therefore remains dispensable as the leader approaches the coalitions in this set. We then show that after the leader secures the votes of coalitions in $M \setminus K(M,q)$ at transfer promises close to 0, now coalitions in $K(M,q)$ become dispensable and therefore the leader can secure their votes at cost close to 0 as well.

We characterized the optimal sequence when the leader bargains with individual members in section 3.2. It is difficult to provide a general characterization when the leader bargains with coalitions with arbitrary voting weights and losses from policy. To illustrate, consider the special case in which a coalition's loss is proportional to its size, that is, $x_i = c_i$. Note that in this case, $X(M,q) = K^{\#}(M,q)$. As a first step, we look for conditions under which the leader pays $(1-\delta)K^{\#}(M,q)$ under the optimal sequence. Consider the following example.

Example 2. *Suppose there are four coalitions, having weights* $2, 2, 6, 7$. *Also suppose that* $q = 8$ *and* $y = 8.5$. *So* $K(M,q) = \{2, 6\}$ *and* $K^{\#}(M,q) = 8 < y$. *An optimal sequence is for the leader to approach a coalition with weight* 2 *first and then approach the coalition with weight* 6. *Note that since the coalition being approached is dispensable at that state, the leader only pays* $(1-\delta)c_i$.

Now suppose everything is the same as before except that the four coalitions have weights $2, 2, 6, 9$. *We still have* $K(M,q) = \{2, 6\}$ *and* $K^{\#}(M,q) = 8 < y$, *but it is no longer an optimal sequence is for the leader to approach a coalition with weight* 2 *first and then approach the coalition with weight* 6. *To see this, note that although a coalition with weight* 2 *is dispensable at the beginning of the game; after it is bought, the coalition with weight* 6 *is not dispensable and therefore the leader has to pay* 6 *in order to has its votes. The optimal sequence in this case is for the leader to approach coalition with* 9 *votes first and pays* $(1-\delta)9$.

One sufficient condition for the leader to pay $(1-\delta)K^{\#}(M,q)$ under the optimal sequence is as follows. If there exists $K(M,q) \in \mathcal{K}(M,q)$ such that $K^{max}(M,q)$ is dispensable at the beginning of the game, then an optimal sequence is for the leader to approach coalitions in $K(M,q)$ is descending order. But as the example above shows, this is not a necessary condition.

6 Appendix

6.1 Proof of Proposition 1

Fix state (S, r, t). We first prove part (a). Assume $y - t > \sum_{j \in S^r} x_j$. Suppose, towards a contradiction, that there exists an equilibrium in $\Gamma(S, r, t)$ in which the policy does not pass. The leader's payoff in this equilibrium is 0. We next show that the leader has a strictly profitable deviation. Note that it is optimal for any member i to accept an offer greater than x_i. Consider the strategy of approaching members in S in a descending order, starting with the member with the largest index in S^r first, offering zero transfer to the first $|S| - r$ members (no matter what the history is) and then offering $x_i + \varepsilon$ to each of the remaining r member (no matter what the history is). Since the last r members accept the offers, policy passes and the leader's payoff is $y - t - \sum_{j \in S^r} x_j - r\varepsilon > 0$ for $\varepsilon > 0$ sufficiently low. Hence, the leader has a profitable deviation, a contradiction.

We next prove part (b) by induction. Assume $y - t < \sum_{j \in S^r} x_j$. First consider $|S| = r$. Suppose, towards a contradiction, that there exists an equilibrium in which the policy passes. Since $|S| = r$, each member $i \in S$ accepts the leader's offer in this equilibrium. Since any member i's rejection leads to the failure of the policy passing, the equilibrium payoff of any member $i \in S$ is nonnegative. Hence, the leader must offer each member i at least x_i, which implies that the leader's payoff in this equilibrium is no higher than $y - t - \sum_{j \in S} x_j < 0$. Since the leader receives a payoff of 0 if she stops immediately in $\Gamma(S, r, t)$, it follows that she has a strictly profitable deviation, a contradiction. Hence, the policy doe not pass in any equilibrium.

Next, suppose that part (b) holds for $|S| - r \leq k$ where $0 \leq k < |S|$. We prove that it also holds for $|S| - r = k + 1$. Suppose, towards a contradiction, that there exists an equilibrium in $\Gamma(S, r, t)$ in which the policy passes. Suppose in this equilibrium, the leader approaches member i in the first period of $\Gamma(S, r, t)$. Note that given the induction hypothesis, if member i rejects the leader's offer, then the policy does not pass in any equilibrium in the resulting subgame $\Gamma(S_{-i}, r, t)$ since $y - t < \sum_{j \in S^r_{-i}} x_j$. Given that the policy passes in equilibrium in $\Gamma(S, r, t)$, the transfer t_i offered to i has to be such that $\delta^r(-x_i + t_i) \geq 0$, that is, $t_i \geq x_i$. Note that in the subgame that follows member i's acceptance, $\Gamma(S_i, r - 1, t + t_i)$, we have $y - t - t_i \leq y - t - x_i$. Since $y - t < \sum_{j \in S^r} x_j$, it follows that $y - t - x_i < \sum_{j \in S^{r-1}_{-i}} x_j$ and therefore $y - t - t_i < \sum_{j \in S^{r-1}_{-i}} x_j$. By the induction hypothesis, the policy does not pass in any equilibrium in $\Gamma(S_{-i}, r - 1, t + t_i)$, a contradiction. Hence, part (b) holds. ■

6.2 Proof of Proposition 2

Given a set of members S, let $(i)_S$ be the member with the ith lowest loss among members in S, that is, $x_{(i)_S} \leq x_{(i+1)_S}$. We prove the following lemma. Also, let S^r denote the set of members in S with the r lowest losses.

Lemma 3. *In the transfer-promise game, consider subgame $\Gamma(S, r, t)$ with $r < |S|$ and $y - t > \sum_{j \in S^r} x_j$. There exists $\bar{\delta} < 1$ such that for $\delta > \bar{\delta}$, the following results hold generically:*
(a) In any equilibrium, only r members are approached and each accepts the leader's offer.

(b) Suppose the leader offer t_i to member $i \in S$ in state (S, r, t). If member i is indispensable, then his equilibrium strategy is to accept t_i if and only if $t_i \geq x_i$; if member i is dispensable, then his equilibrium strategy is to accept t_i if and only if $t_i \geq (1-\delta)x_i$.
(c) If $y - t > \sum_{j=1}^{r-1} x_{(j)_S} + x_{(r+1)_S}$, then (i) there exists an equilibrium in which the leader approaches $i \in S^r$ in descending order; (ii) in any equilibrium, the leader approaches member $i \in S^r$ and offers $t_i = (1-\delta)x_i$.
(d) If $y - t < \sum_{j=1}^{r-1} x_{j(S)} + x_{(r+1)_S}$, then (i) there exists an equilibrium in which the leader first approaches member $(r+1)_S$ and then approaches $i \in S^{r-1}$ in descending order; (ii) in any equilibrium, the leader first approaches member $(r+1)_S$ and offer $(1-\delta)x_{(r+1)_S}$ and then approaches member $i \in S^{r-1}$ and offers $t_i = (1-\delta)x_i$.

We prove the lemma by induction.

First step: we show that the results hold for $|S| = 2$, $r = 1$ and any t. Suppose the leader offer t_i to member $i \in S$. If i is dispensable, then his payoff equals $t_i - x_i$ by accepting the offer and his payoff equals $-\delta x_i$ by rejecting the offer since by Proposition 1 the policy passes in the subgame following the rejection. Hence, he accepts the offer if and only if $t_i - x_i \geq -\delta x_i$, that is, $t_i \geq (1-\delta)x_i$. If i is indispensable, then his payoff equals $t_i - x_i$ by accepting the offer and his payoff equals 0 by rejecting the offer since by Proposition 1 the policy does not passe in the subgame following the rejection. Hence, he accepts the offer if and only if $t_i \geq x_i$. So part (b) holds.

Now we turn to part (c). Suppose $y - t > x_{(2)_S}$, which implies that member $(1)_S$ is dispensable in state (S, r, t). By part (b), he accepts τ if and only if $\tau \geq (1-\delta)x_{(1)_S}$. Since $(1-\delta)x_{(1)_S}$ is the lowest payment the leader has to make for the policy to pass, it follows that in the unique equilibrium, the leader offers $\tau = (1-\delta)x_{(1)_S}$ to member $(1)_S$, which is accepted and the policy passes.

Now we turn to part (d). Suppose $y - t < x_{(2)_S}$, which implies that member $(1)_S$ is indispensable in state (S, r, t). By part (b), he accepts τ if and only if $\tau \geq x_{(1)_S}$. But since $y - t > x_{(1)_S}$, member $(2)_S$ is dispensable in state (S, r, t) and therefore accepts τ if and only if $\tau \geq (1-\delta)x_{(2)_S}$. For δ sufficiently high, we have $(1-\delta)x_{(2)_S} < x_{(1)_S}$. Hence, $(1-\delta)x_{(2)_S}$ is the lowest payment the leader has to make for the policy to pass, it follows that in the unique equilibrium, the leader offers $\tau = (1-\delta)x_{(2)_S}$ to member $(2)_S$, which is accepted and the policy passes. Note that no matter $y - t > x_{(2)_S}$ or $y - t < x_{(2)_S}$, only one member is approached and he accepts the leader's offer. Hence part (a) holds.

Second step: we show that for any $r < |S|$ and any t, if the results hold for $|S| \leq k$, then they hold for $|S| = k + 1$. Suppose the leader offer t_i to member $i \in S$. If i is dispensable, then his payoff equals $(t_i - x_i)\delta^{r-1}$ by accepting the offer and his payoff equals $-\delta^r x_i$ by rejecting the offer since by Proposition 1 and the induction hypothesis the policy passes in the subgame following the rejection in r periods. Hence, he accepts the offer if and only if $(t_i - x_i)\delta^{r-1} \geq -\delta^r x_i$, that is, $t_i \geq (1-\delta)x_i$. If i is indispensable, then his payoff equals $t_i - x_i$ by accepting the offer and his payoff equals 0 by rejecting the offer since by Proposition 1 the policy does not passe in the subgame following the rejection. Hence, he accepts the offer if and only if $t_i \geq x_i$. So part (b) holds.

We next turn to part (c). Suppose $y - t > \sum_{j=1}^{r-1} x_{(j)_S} + x_{(r+1)_S}$, which implies that member $(r)_S$ is dispensable in state (S, r, t). By part (b), he accepts τ if and only if $\tau \geq (1-\delta)x_{(r)_S}$. Note that if δ sufficiently high, then in state $(S_{-r}, t + (1-\delta)x_{(r)_S}, r-1)$, that is, the state after the leader secures the vote of member $(r)_S$ by making the transfer promise $(1-\delta)x_{(r)_S}$, member $(r-1)_{S_{-r}}$ is dispensable. By the induction hypothesis, in state $(S_{-r}, t+(1-\delta)x_{(r)_S}, r-1)$, there exists an equilibrium in which the leader approaches $i \in S^{r-1}$ in descending order, offering each $(1-\delta)x_i$ and the policy passes. Since $(1-\delta)\sum_{j=1}^{r} x_{(j)_S}$ is the lowest total offer that the leader has to make for the policy to pass, it follows that it is an equilibrium for the leader to approach member $(r)_S$ in state (S, r, t) and then approach the members in S_{-r} in descending order, and in any equilibrium, the leader approaches member $i \in S^r$ and offers $t_i = (1-\delta)x_i$. So part (c) holds.

Now we turn to part (d). Suppose $y - t < \sum_{j=1}^{r-1} x_{j(S)} + x_{(r+1)_S}$, which implies that member $(r)_S$ is indispensable in state (S, r, t). By part (b), he accepts τ if and only if $\tau \geq x_{(r)_S}$. But since $y - t > \sum_{j \in S^r} x_j$, member $(r+1)_S$ is dispensable in state (S, r, t) and therefore accepts τ if and only if $\tau \geq (1-\delta)x_{(r+1)_S}$. Note that if δ sufficiently high, then in state $(S \setminus \{r+1)_S\}, t+(1-\delta)x_{(r+1)_S}, r-1)$, that is, the state after the leader secures the vote of member $(r+1)_S$ by making the offer $(1-\delta)x_{(r+1)_S}$, member $(r-1)_S$ is dispensable. By the induction hypothesis, in state $(S \setminus \{r+1)_S\}, t+(1-\delta)x_{(r+1)_S}, r-1)$, there exists an equilibrium in which the leader approaches $i \in S^{r-1}$ in descending order, offering each $(1-\delta)x_i$, and the policy passes. Note that for δ sufficiently high, we have $(1-\delta)x_{(r+1)_S} < x_{(r)_S}$. It follows that $(1-\delta)[x_{(r+1)_S} + \sum_{j=1}^{r-1} x_{(j)_S}]$ is the lowest total offer the leader has to make for the policy to pass. Hence, it is an equilibrium for the leader to approach member $(r+1)_S$ in state (S, r, t) and then approach the members in $S \setminus \{r+1)_S\}$ in descending order, and in any equilibrium, the leader first approaches member $(r+1)_S$ and then approaches member $i \in S^{r-1}$ and offers $t_i = (1-\delta)x_i$. Note that no matter $y - t > \sum_{j=1}^{r-1} x_{(j)_S} + x_{(r+1)_S}$ or $y - t < \sum_{j=1}^{r-1} x_{(j)_S} + x_{(r+1)_S}$, only r members are approached and he accepts the leader's offer. Hence part (a) holds. ■

6.3 Proof of all results in Section 4

The results in Section 4 follow from Lemmas 4, 5, 6, 7 and 8 and from Proposition 12.

Throughout, let $\mathcal{D}^W = \{(S, r) \in 2^N \times \mathbb{Z} | r \leq n\}$, let $\mathcal{D}^G = \{(S, r) \in 2^N \times \mathbb{Z} | S \neq \varnothing \wedge 1 \leq r \leq |S|\}$, for any $S \in 2^N \setminus \varnothing$, let $S' = S \setminus \{\max S\}$, with the convention that $\varnothing' = \varnothing$, and let $\mathcal{L} = \{\sum_{j \in S} x_j | S \in 2^N\} \cup \{\infty\}$. For any $S \in 2^N \setminus \varnothing$, for any $k \in \{1, \ldots, |S|\}$, let (k) be the member with the k-th smallest index in S.[16] That is, $S = \{(k) | k \in \{1, \ldots, |S|\}\}$ and the set of x_is of the members in S is $\{x_{(k)} | k \in \{1, \ldots, |S|\}\}$. Because $x_i \leq x_{i+1}$ $\forall i \in N \setminus \{n\}$, we have, $\forall S \in 2^N \setminus \varnothing$ and $\forall k \in \{1, \ldots, |S|-1\}$, $x_{(k)} \leq x_{(k+1)}$. Use $(k)_S$ instead of (k) only when the underlying S needs to be made explicit.

Let $\bar{\delta} = \max\{\bar{\delta}_a, \bar{\delta}_b, \bar{\delta}_c\}$, where

(*a*) $\bar{\delta}_a$ ensures that, for any $x \in \mathcal{L}$ such that $y - x > 0$, $y - x - nx_n(1-\delta) > 0$, that is $\bar{\delta}_a = 1 - \frac{\min_{x \in \mathcal{L}, x < y}(y-x)}{nx_n}$,

(*b*) $\bar{\delta}_b$ ensures that $nx_n(1-\delta) \leq x_1$, that is, $\bar{\delta}_b = 1 - \frac{x_1}{nx_n}$, and

(*c*) $\bar{\delta}_c$ ensures that, for any $x, x' \in \mathcal{L}$ with $x' < x$, $x' + nx_n(1-\delta) \leq x$, that is, $\bar{\delta}_c = 1 - \frac{\min_{x, x' \in \mathcal{L}, x' < x}(x-x')}{nx_n}$.

[16] Formally, given $S \in 2^N \setminus \varnothing$ and $k \in \{1, \ldots, |S|\}$, (k) is defined recursively as $(k) = \min S$ if $k = 1$ and $(k) = \min\{S \setminus \cup_{i=1}^{k-1}(i)\}$ if $k \geq 2$.

Note that because $\min_{x\in\mathcal{L},x<y}(y-x)>0$, $x_1>0$ and $\min_{x,x'\in\mathcal{L},x'<x}(x-x')>0$, we have $\bar{\delta}<1$.

Define $W:\mathcal{D}^W\to\mathbb{R}\cup\{\infty\}$ as follows. $W(S,r)=0$ for any $(S,r)\in\mathcal{D}^W\setminus\mathcal{D}^G$ with $r\le 0$, $W(S,r)=\infty$ for any $(S,r)\in\mathcal{D}^W\setminus\mathcal{D}^G$ with $r>|S|$, and, for any $(S,r)\in\mathcal{D}^G$,

$$W(S,r)=\min_{T\in 2^S}\max\left\{\sum_{j\in T}x_j, W((S\setminus T)',r-|T|)\right\}. \tag{3}$$

Define $\Pi:\mathcal{D}^G\times\mathbb{R}_{++}\to\mathbb{R}$ as follows

$$\Pi(S,r,y)=\min_{T\in 2^S}\sum_{j\in T}x_j \text{ s.t. } y>W((S\setminus T)',r-|T|). \tag{4}$$

Definition 6. *A member $i\in S$ in state $(S,r)\in\mathcal{D}^W$ is*

1. *dispensable if $y>W(S\setminus\{i\},r)$,*
2. *indispensable if $y\in(W(S\setminus\{i\},r-1),W(S\setminus\{i\},r))$,*
3. *inconsequential if $y<W(S\setminus\{i\},r-1)$.*

Definition 7. *Fix a profile of strategies and consider the resulting sequence of approached members. We say a member approached in a state in which he is indispensable is tempted and a member approached in a state in which he is dispensable is exploited. Suppose any member is tempted before any member is exploited or vice versa. Then the tempted members form a temptation phase and the exploited members form an exploitation phase.*

Lemma 4. *Consider state $(S,r)\in\mathcal{D}^W$.*

1. *If $(S,r)\in\mathcal{D}^G$, then any T that solves (3) satisfies $|T|\le r$.*
2. *If $|S|\ge 1$, then $W(S,1)=x_{(1)}$ and $W(S,|S|)=\sum_{j\in S}x_j$.*
3. *$W(S,r)\in\mathcal{L}$. If $(S,r)\in\mathcal{D}^G$, then $W(S,r)\in(0,\infty)$.*
4. *$W(S,r)\ge W(S,r-1)$.*
5. *$W(\hat{S},r)\le W(S,r)$ for any $\hat{S}\in 2^N$ such that $|\hat{S}|\ge|S|$ and $\hat{x}_{(i)_{\hat{S}}}\le x_{(i)_S}$ $\forall i\in\{1,\dots,|S|\}$.*
6. *If $|S|\ge 1$, then $W(S,r)\ge W(S\setminus\{i\},r-1)$ $\forall i\in S$.*
7. *If $(S,r)\in\mathcal{D}^G$, then $W(S,r)=W(\hat{S},r)$ for any $\hat{S}\in 2^N$ such that $|\hat{S}|=|S|$ and $\hat{x}_{(i)_{\hat{S}}}=x_{(i)_S}$ $\forall i\in\{1,\dots,r\}$.*
8. *If $(S,r)\in\mathcal{D}^G$, then $W(S,r)\le\sum_{j=1}^r x_{(j)}$.*
9. *If $(S,r)\in\mathcal{D}^G$ and $r\le\frac{|S|+1}{2}$, then $W(S,r)=x_{(r)}$.*
10. *If $(S,r)\in\mathcal{D}^G$ and $x_i=x$ $\forall i\in S$, then $W(S,r)=\left\lceil\frac{r}{|S|-r+1}\right\rceil x=\left\lfloor\frac{|S|}{|S|-r+1}\right\rfloor x$.*

Proof. Part 1: Consider $(S,r)\in\mathcal{D}^G$ and T that solves (3) given (S,r). Suppose, towards a contradiction, $|T|>r$. Because $r-|T|<0$, we have $W((S\setminus T)',r-|T|)=0$ and hence $W(S,r)=\sum_{j\in T}x_j$. Now consider T'. We have $|T'|=|T|-1$ because $|T|>r$ and

$(S, r) \in \mathcal{D}^G$ jointly imply $T \neq \varnothing$. Thus $|T'| \geq r$. Hence $\sum_{j \in T'} x_j < \sum_{j \in T} x_j = W(S, r)$ and $W((S \setminus T')', r - |T'|) = 0$, a contradiction because T solves (3) given (S, r).

Part 2: It suffices to prove that, for any $S \in 2^N \setminus \varnothing$, $W(S, 1) = x_{(1)}$ and $W(S, |S|) = \sum_{j \in S} x_j$. To see the latter, consider $S \in 2^N \setminus \varnothing$. Because $S \in 2^N \setminus \varnothing$, $(S, |S|) \in \mathcal{D}^G$. The claim thus follows because the objective function in (3) evaluated at $(S, |S|)$ and $T = S$ equals $\sum_{j \in S} x_j$ and evaluated at $(S, |S|)$ and any $T \in 2^S \setminus S$ equals ∞. We prove the former by induction on $|S|$. Note that for any $S \in 2^N \setminus \varnothing$, $(S, 1) \in \mathcal{D}^G$. That $W(S, 1) = x_{(1)}$ for any $S \in 2^N \setminus \varnothing$ with $|S| = 1$ follows because $W(S, |S|) = \sum_{j \in S} x_j$ for any $S \in 2^N \setminus \varnothing$ with $|S| \geq 1$. Now suppose that $W(S, 1) = x_{(1)}$ for any $S \in 2^N \setminus \varnothing$ with $|S| \leq k$, where $k \geq 1$. We need to prove that $W(S, 1) = x_{(1)}$ for any $S \in 2^N \setminus \varnothing$ with $|S| = k + 1$. To see this, consider T that solves (3) given (S, r). By Part 1, $|T| \in \{0, 1\}$. If $|T| = 0$, then $W(S, 1) = W(S', 1) = x_{(1)}$, where the second equality follows from the induction hypothesis. If $|T| = 1$, then, because $W((S \setminus U)', 1 - |U|) = 0$ for any $U \in 2^S$ with $|U| = 1$, we have $T = \{\min S\}$ as well as $W(S, 1) = \sum_{j \in T} x_j$.

Part 3: We first prove that $W(S, r) \in \mathcal{L}$ for any $(S, r) \in \mathcal{D}^W$. We proceed by induction on $|S|$. That $W(S, r) \in \mathcal{L}$ for any $(S, r) \in \mathcal{D}^W$ with $|S| = 0$ follows directly from definition of W. Now suppose that $W(S, r) \in \mathcal{L}$ for any $(S, r) \in \mathcal{D}^W$ with $|S| \leq k$, where $k \geq 0$. We need to prove that $W(S, r) \in \mathcal{L}$ for any $(S, r) \in \mathcal{D}^W$ with $|S| = k + 1$. To see this, we have either (i) $r \leq 0$, in which case $W(S, r) = 0 \in \mathcal{L}$, or (ii) $r > |S|$, in which case $W(S, r) = \infty \in \mathcal{L}$, or (iii) $r \in \{1, \ldots, |S|\}$, in which case, given T that solves (3) given (S, r), either $W(S, r) = \sum_{j \in T} x_j \in \mathcal{L}$, or, by the induction hypothesis, $W(S, r) = W((S \setminus T)', r - |T|) \in \mathcal{L}$. We now prove that $W(S, r) < \infty$ for any $(S, r) \in \mathcal{D}^G$. This follows because the objective function in (3) evaluated at $(S, r) \in \mathcal{D}^G$ and $T = S$ equals $\sum_{j \in T} x_j < \infty$. We now prove that $W(S, r) > 0$ for any $(S, r) \in \mathcal{D}^G$. We proceed by induction on $|S|$. That $W(S, r) > 0$ for any $(S, r) \in \mathcal{D}^G$ with $|S| = 1$ follows because $(S, r) \in \mathcal{D}^G$ and $|S| = 1$ imply $r = 1$ and hence, by Part 2, $W(S, r) = x_{(1)} > 0$ for any $(S, r) \in \mathcal{D}^G$ with $|S| = 1$. Now suppose that $W(S, r) > 0$ for any $(S, r) \in \mathcal{D}^G$ with $|S| \leq k$, where $k \geq 1$. We need to prove that $W(S, r) > 0$ for any $(S, r) \in \mathcal{D}^G$ with $|S| = k + 1$. To see this, consider T that solves (3) given (S, r). We have either (i) $T = \varnothing$, in which case $W(S, r) = W(S', r) > 0$ either by the induction hypothesis when $(S', r) \in \mathcal{D}^G$ or directly from definition of W when $(S', r) \notin \mathcal{D}^G$ and hence $r > |S'|$, or (ii) $T \neq \varnothing$, in which case $W(S, r) \geq \sum_{j \in T} x_j > 0$.

Part 4: We proceed by induction on $|S|$. That $W(S, r) \geq W(S, r - 1)$ for any $(S, r) \in \mathcal{D}^W$ with $|S| = 0$ follows directly from definition of W. Now suppose that $W(S, r) \geq W(S, r - 1)$ for any $(S, r) \in \mathcal{D}^W$ with $|S| \leq k$, where $k \geq 0$. We need to prove that $W(S, r) \geq W(S, r - 1)$ for any $(S, r) \in \mathcal{D}^W$ with $|S| = k + 1$. To see this, we have either (i) $r \leq 1$, in which case $W(S, r) \geq \min \mathcal{L} = 0$ by Part 3 and $W(S, r - 1) = 0$, or (ii) $r > |S|$, in which case $W(S, r) = \infty \geq W(S, r - 1)$, or (iii) $r \in \{2, \ldots, |S|\}$, in which case, given T that solves (3) given (S, r), $W(S, r) = \max\{\sum_{j \in T} x_j, W((S \setminus T)', r - |T|)\} \geq \max\{\sum_{j \in T} x_j, W((S \setminus T)', r - 1 - |T|)\} \geq W(S, r - 1)$, where the first inequality follows from the induction hypothesis.

Part 5: We proceed by induction on $|S|$. That $W(\hat{S}, r) \leq W(S, r)$ for any $(S, r) \in \mathcal{D}^W$ with $|S| = 0$ and for any $\hat{S} \in 2^N$ such that $|\hat{S}| \geq |S|$ and $\hat{x}_{(i)_{\hat{S}}} \leq x_{(i)_S}$ $\forall i \in \{1, \ldots, |S|\}$ follows because given $S = \varnothing$ and any $\hat{S} \in 2^N$ we have either (i) $r \leq 0$, in which case $W(S, r) = W(\hat{S}, r) = 0$, or (ii) $r \geq 1$, in which case $W(S, r) = \infty \geq W(\hat{S}, r)$. Now suppose that $W(\hat{S},r) \le W(S,r)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| \le k$, where $k \ge 0$, and for any $\hat{S} \in 2^N$ such that $|\hat{S}| \ge |S|$ and $\hat{x}_{(i)_{\hat{S}}} \le x_{(i)_S}$ $\forall i \in \{1,\dots,|S|\}$. We need to prove that $W(\hat{S},r) \le W(S,r)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| = k+1$ and for any $\hat{S} \in 2^N$ such that $|\hat{S}| \ge |S|$ and $\hat{x}_{(i)_{\hat{S}}} \le x_{(i)_S}$ $\forall i \in \{1,\dots,|S|\}$. To see this, we have either (i) $r \le 0$, in which case $W(S,r) = W(\hat{S},r) = 0$, or (ii) $r > |S|$, in which case $W(S,r) = \infty \ge W(\hat{S},r)$, or (iii) $r \in \{1,\dots,|S|\}$, in which case, given T that solves (3) given (S,r) and $\hat{T} = \{(k)_{\hat{S}} | k \in \{1,\dots,|S|\}, (k)_S \in T\}$, $W(\hat{S},r) \le \max\{\sum_{j\in\hat{T}} \hat{x}_j, W((\hat{S}\setminus\hat{T})', r-\hat{T})\} \le \max\{\sum_{j\in T} x_j, W((S\setminus T)', r-|T|)\} = W(S,r)$, where the second inequality follows from the construction of $\hat{T}$ and from the induction hypothesis.

Part 6: Let $S^* = S \setminus \{\min S\}$ for any $S \in 2^N \setminus \varnothing$. Note that for any $S \in 2^N$ with $|S| \ge 2$, $(S^*)' = (S')^*$. By Part 5, it suffices to prove that $W(S,r) \ge W(S^*, r-1)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| \ge 1$, which we prove by induction on $|S|$. That $W(S,r) \ge W(S^*, r-1)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| = 1$ follows either directly from definition of W when $(S,r) \notin \mathcal{D}^G$ or from $W(S,r) = x_{(1)} > 0$ shown in Part 2 and $W(S^*, r-1) = 0$ when $(S,r) \in \mathcal{D}^G$. Now suppose that $W(S,r) \ge W(S^*, r-1)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| \le k$, where $k \ge 1$. We need to prove that $W(S,r) \ge W(S^*, r-1)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| = k+1$. To see this, we have either (i) $r \le 1$, in which case $W(S,r) \ge \min \mathcal{L} = 0$ by Part 3 and $W(S^*, r-1) = 0$, or (ii) $r > |S|$, in which case $W(S,r) = W(S^*, r-1) = \infty$, or (iii) $r \in \{2,\dots,|S|\}$, in which case, given T that solves (3) given (S,r), either (a) $\min S \in T$, in which case $S \setminus T = S^* \setminus T^*$ and $T^* \in 2^{S^*}$ and hence $W(S,r) = \max\{\sum_{j\in T} x_j, W((S\setminus T)', r-|T|)\} \ge \max\{\sum_{j\in T^*} x_j, W((S^* \setminus T^*)', r-1-|T^*|)\} \ge W(S^*, r-1)$, or (b) $\min S \notin T$ and $|T| \ge r-1$, in which case $W((S^*\setminus T)', r-1-|T|) = 0$ and $T \in 2^{S^*}$ and hence $W(S,r) = \max\{\sum_{j\in T} x_j, W((S\setminus T)', r-|T|)\} \ge \sum_{j\in T} x_j = \max\{\sum_{j\in T} x_j, W((S^*\setminus T)', r-1-|T|)\} \ge W(S^*, r-1)$, or (c) $\min S \notin T$ and $|T| \le r-2$, in which case $|T| \le |S|-2$, $(S\setminus T)^* = S^* \setminus T$ and $T \in 2^{S^*}$ and hence $W(S,r) = \max\{\sum_{j\in T} x_j, W((S\setminus T)', r-|T|)\} \ge \max\{\sum_{j\in T} x_j, W(((S\setminus T)')^*, r-1-|T|)\} = \max\{\sum_{j\in T} x_j, W((S^*\setminus T)', r-1-|T|)\} \ge W(S^*, r-1)$, where the first inequality follows from the induction hypothesis.

Part 7: We proceed by induction on $|S|$. That $W(S,r) = W(\hat{S},r)$ for any $(S,r) \in \mathcal{D}^G$ with $|S| = 1$ and for any $\hat{S} \in 2^N$ such that $|\hat{S}| = |S|$ and $\hat{x}_{(i)_{\hat{S}}} = x_{(i)_S}$ $\forall i \in \{1,\dots,r\}$ follows because, as shown in Part 2, $W(S,1) = x_{(1)_S}$ for any $S \in 2^N \setminus \varnothing$. Now suppose that $W(S,r) = W(\hat{S},r)$ for any $(S,r) \in \mathcal{D}^G$ with $|S| \le k$, where $k \ge 1$, and for any $\hat{S} \in 2^N$ such that $|\hat{S}| = |S|$ and $\hat{x}_{(i)_{\hat{S}}} = x_{(i)_S}$ $\forall i \in \{1,\dots,r\}$. We need to prove that $W(S,r) = W(\hat{S},r)$ for any $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$ and for any $\hat{S} \in 2^N$ such that $|\hat{S}| = |S|$ and $\hat{x}_{(i)_{\hat{S}}} = x_{(i)_S}$ $\forall i \in \{1,\dots,r\}$. To see this, we have either (i) $r = 1$, in which case $W(S,r) = x_{(1)_S} = \hat{x}_{(1)_{\hat{S}}} = W(\hat{S},r)$ by Part 2, or (ii) $r = |S|$, in which case $W(S,r) = \sum_{j\in S} x_j = \sum_{j\in\hat{S}} \hat{x}_j = W(\hat{S},r)$ by Part 2, or (iii) $r \in \{2,\dots,|S|-1\}$, in which case, given T that solves (3) given (S,r), either (a) $|T| = 0$, in which case $W(S,r) = W(S',r) = W(\hat{S}',r) \ge W(\hat{S},r)$, where the second equality follows from the induction hypothesis, or (b) $|T| = r$, in which case $T = \{(k)_S | k \in \{1,\dots,r\}\}$ because $W((S\setminus T)', r-|T|) = 0$ and thus $W(S,r) = \sum_{j=1}^r x_{(j)_S} = \sum_{j=1}^r \hat{x}_{(j)_{\hat{S}}} \ge W(\hat{S},r)$, or (c) $|T| \in \{1,\dots,r-1\}$, in which case, given $\hat{T} = \{(k)_{\hat{S}} | k \in \{1,\dots,r\}, (k)_S \in T\}$, $W(S,r) = \max\{\sum_{j\in T} x_j, W((S\setminus T)', r-|T|)\} = \max\{\sum_{j\in\hat{T}} \hat{x}_j, W((\hat{S}\setminus\hat{T})', r-|\hat{T}|)\} \ge W(\hat{S},r)$, where the second equality follows because, as we show below, it is without loss of generality to assume that $T \subseteq \{(k)_S | k \in \{1, \ldots, r\}\}$, and hence we have $\sum_{j\in T} x_j = \sum_{j\in \hat{T}} \hat{x}_j$ as well as $((S \setminus T)', r - |T|) \in \mathcal{D}^G$ implied by $|T| \in \{1, \ldots, r-1\}$ and $r \leq |S|-1$, $|(S \setminus T)'| = |(\hat{S} \setminus \hat{T})'|$ and $x_{(i)_{(S\setminus T)'}} = \hat{x}_{(i)_{(\hat{S}\setminus\hat{T})'}}$ $\forall i \in \{1, \ldots, r - |T|\}$ and thus, by the induction hypothesis, $W((S \setminus T)', r - |T|) = W((\hat{S} \setminus \hat{T})', r - |\hat{T}|)$. All three subcases of the (iii) case show that $W(S, r) \geq W(\hat{S}, r)$. Swapping S and $\hat{S}$ in the argument shows that $W(\hat{S}, r) \geq W(S, r)$ and hence that $W(S, r) = W(\hat{S}, r)$.

What remains is to show that if T solves (3) given (S, r) with $|S| = k+1$, where $k \geq 1$, and $r \in \{2, \ldots, |S|-1\}$ and if $|T| \in \{1, \ldots, r-1\}$, then there exists another solution of (3), T_a, such that $T_a \subseteq \{(k)_S | k \in \{1, \ldots, r\}\}$ and $|T_a| = |T|$. To see this, let $T_r = \{(k)_S | k \in \{1, \ldots, r\}, (k)_S \in T\}$, $T_{r+1} = \{(k)_S | k \in \{r+1, \ldots, |S|\}, (k)_S \in T\}$, $U_r = \{(k)_S | k \in \{1, \ldots, r\}, (k)_S \in S \setminus T\}$ and $U_{r+1} = \{(k)_S | k \in \{r+1, \ldots, |S|\}, (k)_S \in S \setminus T\}$. Note that because $T \in 2^S$, $S = T_r \cup T_{r+1} \cup U_r \cup U_{r+1}$. If $|T_{r+1}| = 0$ the claim follows by setting $T_a = T$. Hence, suppose that $|T_{r+1}| \geq 1$. Because $|T| = |T_r| + |T_{r+1}|$ and $|T_r| + |U_r| = r$, we have $|U_r| = r - |T| + |T_{r+1}|$ and hence there exists a partition of U_r into U_s and U_m such that $|U_s| = r - |T| \geq 1$, $|U_m| = |T_{r+1}| \geq 1$ and $i < j$ for any $i \in U_s$ and $j \in U_m$. Now construct $T_a = T_r \cup U_m$. By construction $T_a \subseteq \{(k)_S | k \in \{1, \ldots, r\}\}$ and $|T_a| = |T|$. Moreover, we have $\max\{\sum_{j\in T} x_j, W((S \setminus T)', r - |T|)\} \geq \max\{\sum_{j\in T_a} x_j, W((S \setminus T_a)', r - |T_a|)\}$, where the inequality follows from $\sum_{j\in T} x_j \geq \sum_{j\in T_a} x_j$, which holds because $T = T_r \cup T_{r+1}$, $T_a = T_r \cup U_m$ and $U_m \subset U_r$, and from $W((S \setminus T)', r - |T|) = W((S \setminus T_a)', r - |T_a|)$, which holds by the induction hypothesis because $((S \setminus T)', r - |T|) \in \mathcal{D}^G$, $S \setminus T = U_s \cup U_m \cup U_{r+1}$, $S \setminus T_a = U_s \cup T_{r+1} \cup U_{r+1}$, $|U_m| = |T_{r+1}|$, $i < j$ for any $i \in U_s$ and $j \in U_m \cup U_{r+1} \cup T_{r+1}$ and $|U_s| = r - |T| = r - |T_a|$.

Part 8: We proceed by induction on $|S|$. That $W(S, r) \leq \sum_{j=1}^{r} x_{(j)}$ for any $(S, r) \in \mathcal{D}^G$ with $|S| = 1$ follows because, as shown in Part 2, $W(S, r) = x_{(1)}$ for any $(S, r) \in \mathcal{D}^G$ with $|S| = 1$. Now suppose that $W(S, r) \leq \sum_{j=1}^{r} x_{(j)}$ for any $(S, r) \in \mathcal{D}^G$ with $|S| \leq k$, where $k \geq 1$. We need to prove that $W(S, r) \leq \sum_{j=1}^{r} x_{(j)}$ for any $(S, r) \in \mathcal{D}^G$ with $|S| = k + 1$. To see this, we have either (i) $r = |S|$, in which case $W(S, r) = \sum_{j\in S} x_j$ by Part 2, or (ii) $r < |S|$, in which case $W(S, r) \leq W(S', r) \leq \sum_{j=1}^{r} x_{(j)}$, where the inequality follows from the induction hypothesis.

Part 9: We proceed by induction on r. That $W(S, r) = x_{(r)}$ for any $(S, r) \in \mathcal{D}^G$ with $r \leq \frac{|S|+1}{2}$ and with $r = 1$ follows from Part 2. Now suppose that $W(S, r) = x_{(r)}$ for any $(S, r) \in \mathcal{D}^G$ with $r \leq \frac{|S|+1}{2}$ and with $r \leq k$, where $k \geq 1$. We need to prove that $W(S, r) = x_{(r)}$ for any $(S, r) \in \mathcal{D}^G$ with $r \leq \frac{|S|+1}{2}$ and with $r = k+1$. To see this, consider $(S, r) \in \mathcal{D}^G$ with $r \leq \frac{|S|+1}{2}$ and with $r = k+1$. We claim that $T = \{\min S\}$ solves (3) and thus $W(S, r) = \max\{x_{(1)}, W((S \setminus \{\min S\})', r-1)\} = \max\{x_{(1)}, x_{(r)}\} = x_{(r)}$, where the second equality follows from the induction hypothesis. To see that $\max\{\sum_{j\in T} x_j, W((S \setminus T)', r - |T|)\} \geq x_{(r)}$ for any $T \in 2^S$, given $T \in 2^S$ we have either (i) $|T| \geq r$, in which case $\sum_{j\in T} x_j \geq x_{(r)}$, or (ii) $|T| = 0$, in which case, because $r \geq 2$ implies $|S| \geq 3$ and hence $S' \setminus \{\min S\} = (S \setminus \{\min S\})'$, we have $W((S \setminus T)', r - |T|) = W(S', r) \geq W(S' \setminus \{\min S\}, r-1) = W((S \setminus \{\min S\})', r-1) = x_{(r)}$, where the inequality follows from Part 6, or (iii) $|T| \in \{1, \ldots, r-1\}$, in which case either (a) $(k) \in T$ for some $k \in \{r, \ldots, |S|\}$, in which case $\sum_{j\in T} x_j \geq x_{(r)}$, or (b) $(k) \notin T$ $\forall k \in \{r, \ldots, |S|\}$, in which case $W((S \setminus T)', r - |T|) = x_{(r)}$, where the equality follows from the induction hypothesis, which we can invoke because for any $T \in 2^S$ with $|T| \in \{1, \ldots, r-1\}$, we have $((S \setminus T)', r - |T|) \in \mathcal{D}^G$ and $r - |T| \leq \frac{|(S\setminus T)'|+1}{2}$. The former follows from $|(S \setminus T)'| = |S| - 1 - |T| \geq (2r-1) - 1 - |T| \geq r - |T| \geq 1$, where the second inequality uses $r = k+1 \geq 2$. The latter is equivalent to $r + \frac{1-|T|}{2} \leq \frac{|S|+1}{2}$ and follows from $|T| \geq 1$.

Part 10: We first prove that for any $(S, r) \in \mathcal{D}^G$, we have $\left\lceil \frac{r}{|S|-r+1} \right\rceil = \left\lfloor \frac{|S|}{|S|-r+1} \right\rfloor$. To see this, $(S, r) \in \mathcal{D}^G$ implies $|S| \geq 1$ and $r \in \{1, \ldots, |S|\}$. Thus $|S| - r + 1 \geq 1$ and $0 < \frac{r}{|S|-r+1} \leq \frac{|S|}{|S|-r+1}$. It thus suffices to prove that there exists unique $m \in \mathbb{N}$ such that $\frac{r}{|S|-r+1} \leq m \leq \frac{|S|}{|S|-r+1}$. To see that m exists, if $\frac{r}{|S|-r+1} \notin \mathbb{N}$, then for the smallest integer larger than $\frac{r}{|S|-r+1}$, m', we have $\frac{r+i}{|S|-r+1} = m'$, where $i \in \{1, \ldots, |S| - r\}$. Thus $\frac{|S|}{|S|-r+1} = \frac{|S|-r-i}{|S|-r+1} + m' \geq m'$. That m is unique follows from $\frac{|S|}{|S|-r+1} - \frac{r}{|S|-r+1} < 1$.

We now prove that if $(S, r) \in \mathcal{D}^G$ and $x_i = x \; \forall i \in S$, then $W(S, r) = \left\lceil \frac{r}{|S|-r+1} \right\rceil x = \left\lfloor \frac{|S|}{|S|-r+1} \right\rfloor x$. We proceed by induction on $|S| - r$. Suppose $x_i = x \; \forall i \in S$. That $W(S, r) = \left\lceil \frac{r}{|S|-r+1} \right\rceil x = \left\lfloor \frac{|S|}{|S|-r+1} \right\rfloor x$ for any $(S, r) \in \mathcal{D}^G$ with $|S| - r = 0$ follows because, by Part 2, $W(S, |S|) = \sum_{j \in S} x_j$ for any $S \in 2^N \setminus \varnothing$. Now suppose that $W(S, r) = \left\lceil \frac{r}{|S|-r+1} \right\rceil x = \left\lfloor \frac{|S|}{|S|-r+1} \right\rfloor x$ for any $(S, r) \in \mathcal{D}^G$ with $|S| - r \leq k$, where $k \geq 0$. We need to prove that $W(S, r) = \left\lceil \frac{r}{|S|-r+1} \right\rceil x = \left\lfloor \frac{|S|}{|S|-r+1} \right\rfloor x$ for any $(S, r) \in \mathcal{D}^G$ with $|S| - r = k+1$. To see this, because $x_i = x \; \forall i \in S$ and by Part 1, (3) simplifies to $W(S, r) = \min_{t \in \{0, \ldots, r\}} \max\{tx, W(\cup_{i=1}^{|S|-t-1}\{i\}, r - t)\} = x \min_{t \in \{0, \ldots, r\}} \max\{t, \lceil \frac{r-t}{|S|-r} \rceil\}$, where the second equality follows from the induction hypothesis when $r - t \geq 1$ and directly from definition of W when $r - t = 0$. It thus suffices to prove that $\min_{t \in \{0, \ldots, r\}} \max\{t, \lceil \frac{r-t}{|S|-r} \rceil\} = \lceil \frac{r}{|S|-r+1} \rceil = \lfloor \frac{|S|}{|S|-r+1} \rfloor$. The structure of the problem implies that $\min_{t \in \{0, \ldots, r\}} \max\{t, \lceil \frac{r-t}{|S|-r} \rceil\} = t^*$, where $t^* \in \mathbb{N}$ is the largest solution to the problem. In order to derive t^* we need to consider two cases: either $t = \lceil \frac{r-t}{|S|-r} \rceil$ for some $t \in \{0, \ldots, r\}$ or $t \neq \lceil \frac{r-t}{|S|-r} \rceil \; \forall t \in \{0, \ldots, r\}$. In the former case $t^* = \lceil \frac{r-t^*}{|S|-r} \rceil = \frac{r-t^*+i}{|S|-r}$, where $i \in \{0, \ldots, |S| - r - 1\}$, and thus $t^* = \frac{r+i}{|S|-r+1} = \lceil \frac{r}{|S|-r+1} \rceil$, where the second equality follows from $t^* \in \mathbb{N}$ and $i \leq |S| - r - 1$. In the latter case $t^* = \frac{r-t^*}{|S|-r} + 1$, and thus $t^* = \frac{|S|}{|S|-r+1} = \lfloor \frac{|S|}{|S|-r+1} \rfloor$. ∎

Lemma 5. *Consider state $(S, r) \in \mathcal{D}^G$ and $y > 0$.*

1. *(4) has a solution. $|T| \leq r$ for any T that solves (4).*
2. *If $r = |S|$, then $T = S$ is the unique solution to (4).*
3. *If $y > W(S, r)$, then $y - \Pi(S, r, y) > 0$.*
4. *If $y > W(S', r)$, then $T = \varnothing$ is the unique solution to (4).*
5. *If $y < W(S', r)$, then $T \neq \varnothing$ for any T that solves (4).*
6. *If $r < |S|$ and $y > \sum_{j=1}^{r} x_{(j)}$, then $T = \varnothing$ is the unique solution to (4).*
7. *If $r < \frac{|S|+1}{2}$ and $y > x_{(r)}$, then $T = \varnothing$ is the unique solution to (4).*
8. *If $x \neq y$ for any $x, y \in \{\sum_{j \in T} x_j | T \in 2^N\}$, then (4) has a unique solution.*
9. *If $r < |S|$, then $\forall i \in \{r+1, \ldots, |S|\}$, if $x_{(r)} < x_{(i)}$, then $(i) \notin T$ for any T that solves (4).*

Proof. Fix $(S, r) \in \mathcal{D}^G$ and $y > 0$ throughout. Part 1: (4) has a solution because 2^S is a finite set and any $T \in 2^S$ such that $|T| = r$ is admissible in (4) because $W((S \setminus T)', r - |T|) = 0$. To see that $|T| \leq r$ for any T that solves (4), consider T that solves (4) and suppose, towards a contradiction, that $|T| > r$. When $r = |S|$, we have $|T| > |S|$, a contradiction to $T \in 2^S$. When $r < |S|$, consider any $T_a \subseteq T$ such that $|T_a| = r$. Because $|T| > r$, such T_a exists. We have $T_a \in 2^S$, $\sum_{j \in T} x_j > \sum_{j \in T_a} x_j$ and $W((S \setminus T_a)', r - |T_a|) = 0$, which is a contradiction because T solves (4).

Part 2: Because $r = |S|$ we have, for any $T \in 2^S \setminus S$, $|(S \setminus T)'| = |S| - |T| - 1 = r - |T| - 1 > r - |T| \geq 1$ and hence $W((S \setminus T)', r - |T|) = \infty$. For $T = S$, we have $W((S \setminus T)', r - |T|) = W(\varnothing, 0) = 0$ and hence S is the unique solution to (4).

Part 3: When $r = |S|$, we have $W(S, r) = \sum_{j \in S} x_j$ by Lemma 4 part 2 and $\Pi(S, r, y) = \sum_{j \in S} x_j$ by Part 2. When $r < |S|$, consider $T \in 2^S$ that solves (3) given (S, r). Because $y > W(S, r) = \max\{\sum_{j \in T} x_j, W((S \setminus T)', r - |T|)\}$, we have $y > W((S \setminus T)', r - |T|)$ and $y > \sum_{j \in T} x_j$. The former inequality implies $\Pi(S, r, y) \leq \sum_{j \in T} x_j$ and hence the latter inequality implies $y > \Pi(S, r, y)$.

Parts 4 and 5: $T = \varnothing$ solves (4) when $y > W(S', r)$ because $W((S \setminus T)', r - |T|) = W(S', r)$ when $T = \varnothing$ and is the unique solution because $\sum_{j \in T_a} x_j > 0$ for any $T_a \in 2^S \setminus \varnothing$. $T \neq \varnothing$ for any T that solves (4) when $y < W(S', r)$ because $T = \varnothing$ is not admissible in (4) when $y < W(S', r)$.

Part 6: Suppose $r < |S|$ and $y > \sum_{j=1}^{r} x_{(j)}$. Because $r < |S|$, we have $r \leq |S'|$ and thus $(S', r) \in \mathcal{D}^G$. Hence, by Lemma 4 part 8, $\sum_{j=1}^{r} x_{(j)} \geq W(S', r)$. Thus $y > W(S', r)$ and hence, by Part 4, $T = \varnothing$ is the unique solution to (4).

Part 7: Suppose $r < \frac{|S|+1}{2}$ and $y > x_{(r)}$. Because $r < \frac{|S|+1}{2}$, we have $r \leq \frac{|S|}{2}$ and thus, by Lemma 4 part 9, $W(S, r) = W(S', r) = x_{(r)}$. Thus $y > W(S', r)$ and hence, by Part 4, $T = \varnothing$ is the unique solution to (4).

Part 8: Suppose, towards a contradiction, that T_a and T_b are two distinct solutions to (4). Because $x \neq y$ for any $x, y \in \{\sum_{j \in T} x_j | T \in 2^N\}$, $T_a \in 2^S$, $T_b \in 2^S$ and $S \subseteq N$, we have $\sum_{j \in T_a} x_j \neq \sum_{j \in T_b} x_j$, a contradiction.

Part 9: Suppose $r < |S|$ and consider T that solves (4). Suppose, towards a contradiction, that for some $i \in \{r+1, \ldots, |S|\}$ we have $x_{(r)} < x_{(i)}$ and $(i) \in T$. Note that $|T| \leq r$ from Part 1 and $r < |S|$ imply $S \setminus T \neq \varnothing$. There are two cases to consider. In each case we construct $T_a \in 2^S$ such that $\sum_{j \in T} x_j > \sum_{j \in T_a} x_j$ and $W((S \setminus T)', r - |T|) = W((S \setminus T_a)', r - |T_a|)$ establishing a contradiction to T solving (4).

Case 1: $\max S \setminus T \leq (r)$. Because $\max S \setminus T \leq (r)$, we have $\max S \setminus T \leq (r) < (i)$ and hence $x_{\max S \setminus T} \leq x_{(r)} < x_{(i)}$. Let $T_a = (T \cup \{\max S \setminus T\}) \setminus \{(i)\} \in 2^S$. By construction, $\sum_{j \in T} x_j > \sum_{j \in T_a} x_j$. Moreover, $|T| = |T_a|$ and $(S \setminus T_a)' = (((S \setminus T) \setminus \{\max S \setminus T\}) \cup \{(i)\})' = (S \setminus T)'$ and hence $W((S \setminus T)', r - |T|) = W((S \setminus T_a)', r - |T_a|)$.

Case 2: $\max S \setminus T > (r)$. When $|T| = r$, $T_a = \{(k)_S | k \in \{1, \ldots, r\}\}$ has the desired properties. When $|T| \in \{1, \ldots, r-1\}$, let $T_r = \{(k)_S | k \in \{1, \ldots, r\}, (k)_S \in T\}$ and $U_r = \{(k)_S | k \in \{1, \ldots, r\}, (k)_S \in S \setminus T\}$. By construction, $|T_r| + |U_r| = r$. Moreover, because $(i) \in T$ and $i \geq r+1$, $|T| \geq |T_r| + 1$ and hence $|U_r| = r - |T_r| \geq r - |T| + 1$. Thus, there exists a partition of U_r into U_s and U_m such that $|U_s| = r - |T| \geq 1$, $|U_m| \geq 1$ and $x_j \leq x_{j'}$ for any $j \in U_s$ and $j' \in U_m$. Let $T_a = (T \cup \{i'\}) \setminus \{(i)\}$ for some $i' \in U_m$. By construction, $T_a \in 2^S$. Moreover, because $x_{(i)} > x_{(r)}$ and $(r) \geq i'$, we have $x_{(i)} > x_{(r)} \geq x_{i'}$ and thus $\sum_{j \in T} x_j > \sum_{j \in T_a} x_j$. Finally, $((S \setminus T)', r - |T|) \in \mathcal{D}^G$,

$S \setminus T = U_s \cup U_m \cup Z \cup \{\max S \setminus T\}$, $S \setminus T_a = U_s \cup (U_m \setminus \{i'\}) \cup Z \cup \{\max S \setminus T, (i)\}$, $j < j'$ for any $j \in U_s$ and $j' \in U_m \cup Z \cup \{\max S \setminus T, (i)\}$ and $|U_s| = r - |T| = r - |T_a|$ and thus, by Lemma 4 part 7, $W((S \setminus T)', r - |T|) = W((S \setminus T_a)', r - |T_a|)$. ■

Lemma 6. *Consider state $(S, r) \in \mathcal{D}^G$ and $y > 0$.*

1. $\Pi(S, r, y) \geq \Pi(S, r - 1, y)$ *if* $r \geq 2$.
2. $\Pi(S, r, y) \geq \Pi(\hat{S}, r, y)$ *for any* $\hat{S} \in 2^N$ *such that* $|\hat{S}| \geq |S|$ *and* $\hat{x}_{(i)_{\hat{S}}} \leq x_{(i)_S}$ $\forall i \in \{1, \ldots, |S|\}$.
3. $\Pi(S, r, y) \geq \Pi(S \setminus \{i\}, r - 1, y)$ *for any* $i \in S$ *if* $r \geq 2$.
4. $\Pi(S, r, y) \geq \Pi(S, r, y')$ *if* $y' > y$.

Proof. Fix $(S, r) \in \mathcal{D}^G$ and $y > 0$ throughout. Consider T that solves (4). We have $y > W((S \setminus T)', r - |T|)$. Part 1: From Lemma 4 part 4, we have $W((S \setminus T)', r - |T|) \geq W((S \setminus T)', r - 1 - |T|)$ and hence $y > W((S \setminus T)', r - 1 - |T|)$. Because $(S, r - 1) \in \mathcal{D}^G$, $\Pi(S, r, y) \geq \Pi(S, r - 1, y)$.

Part 2: Consider $\hat{S} \in 2^N$ such that $|\hat{S}| \geq |S|$ and $\hat{x}_{(i)_{\hat{S}}} \leq x_{(i)_S}$ $\forall i \in \{1, \ldots, |S|\}$. Let $\hat{T} = \{(k)_{\hat{S}} | k \in \{1, \ldots, |S|\}, (k)_S \in T\} \in 2^{\hat{S}}$. From Lemma 4 part 5, we have $W((S \setminus T)', r - |T|) \geq W((\hat{S} \setminus \hat{T})', r - |\hat{T}|)$ and hence $y > W((\hat{S} \setminus \hat{T})', r - |\hat{T}|)$. Moreover, because $\hat{x}_{(i)_{\hat{S}}} \leq x_{(i)_S}$ $\forall i \in \{1, \ldots, |S|\}$, we have $\sum_{j \in T} x_j \geq \sum_{j \in \hat{T}} \hat{x}_j$. Because $(\hat{S}, r) \in \mathcal{D}^G$, $\Pi(S, r, y) \geq \Pi(\hat{S}, r, y)$.

Part 3: Consider $i \in S$. We have either $i \in T$ or $i \notin T$. When $i \in T$, set $T_a = T \setminus \{i\} \in 2^{S \setminus \{i\}}$. We have $|T| = |T_a| + 1$ and $S \setminus T = (S \setminus \{i\}) \setminus T_a$. Thus $W((S \setminus T)', r - |T|) = W(((S \setminus \{i\}) \setminus T_a)', r - 1 - |T_a|)$ and hence $y > W(((S \setminus \{i\}) \setminus T_a)', r - 1 - |T_a|)$. Moreover, $\sum_{j \in T} x_j \geq \sum_{j \in T_a} x_j$. Because $(S \setminus \{i\}, r - 1) \in \mathcal{D}^G$, $\Pi(S, r, y) \geq \Pi(S \setminus \{i\}, r - 1, y)$. When $i \notin T$, we have either $|S \setminus T| = 1$ or $|S \setminus T| \geq 2$. In the former case $T \cup \{i\} = S$ so that $(S \setminus T)' = ((S \setminus \{i\}) \setminus T)' = \varnothing$ and Lemma 4 part 4 imply $W((S \setminus T)', r - |T|) = W(((S \setminus \{i\}) \setminus T)', r - 1 - |T|)$ and hence $y > W(((S \setminus \{i\}) \setminus T)', r - 1 - |T|)$. In the latter case either $((S \setminus \{i\}) \setminus T)' = (S \setminus T)' \setminus \{\max(S \setminus T)'\}$ when $i = \max S \setminus T$ or $((S \setminus \{i\}) \setminus T)' = (S \setminus T)' \setminus \{i\}$ when $i < \max S \setminus T$ and Lemma 4 part 6 imply $W((S \setminus T)', r - |T|) \geq W(((S \setminus \{i\}) \setminus T)', r - 1 - |T|)$ and hence $y > W(((S \setminus \{i\}) \setminus T)', r - 1 - |T|)$. Moreover, because $i \notin T$, $T \in 2^{S \setminus \{i\}}$. Because $(S \setminus \{i\}, r - 1) \in \mathcal{D}^G$, $\Pi(S, r, y) \geq \Pi(S \setminus \{i\}, r - 1, y)$.

Part 4: We have $y' > W((S \setminus T)', r - |T|)$ because $y' > y$. Thus $\Pi(S, r, y) \geq \Pi(S, r, y')$. ■

Lemma 7. *If $i \in S$ is dispensable at $(S, r) \in \mathcal{D}^W$, then any $j \in S$ with $j > i$ is dispensable at (S, r). If $i \in S$ is dispensable at $(S, r) \in \mathcal{D}^W$, then, $\forall j \in S \setminus \{i\}$, i is dispensable at $(S \setminus \{j\}, r - 1)$ and j is dispensable at $(S \setminus \{i\}, r - 1)$.*

Proof. To prove the first sentence, using Definition 6, it suffices to prove that, for any $i, j \in S$ with $j > i$, if $y > W(S \setminus \{i\}, r)$, then $y > W(S \setminus \{j\}, r)$, which holds because, by Lemma 4 part 5, we have $W(S \setminus \{i\}, r) \geq W(S \setminus \{j\}, r)$. To prove the second sentence, using Definition 6, it suffices to prove that, for any $i, j \in S$ with $j \neq i$, if $y > W(S \setminus \{i\}, r)$, then $y > W(S \setminus \{i, j\}, r - 1)$, which holds because, by Lemma 4 part 6, we have $W(S \setminus \{i\}, r) \geq W(S \setminus \{i, j\}, r - 1)$. ■

Lemma 8. *Consider state $(S, r) \in \mathcal{D}^G$.*

1. $W(S,r) \le \frac{\sum_{j\in S} x_j}{|S|-r+1}$.
2. *Suppose* $x_{(i)} = a$ *for* $i \in \{1,\ldots,n_a\}$, $x_{(i)} = b$ *for* $i \in \{n_a+1,\ldots,n_a+n_b\}$, *where* $a, b > 0$, $n_a, n_b \in \{0,\ldots,|S|\}$, $n_a+n_b = |S|$ *and* $b > 2|S|a$. *Let* $\mathbb{I}_{in}(x) = 1$ *if* $x \in \mathbb{Z}$ *or* $x \le 0$ *and* $\mathbb{I}_{in}(x) = 0$ *otherwise. Then* $W(S,r) = b\lceil \frac{n_b-(|S|-r)}{|S|-r+1}\rceil + a\left(\lceil \frac{n_a-|S|}{|S|-r+1}\rceil + \lceil \frac{r}{|S|-r+1}\rceil\right)\mathbb{I}_{in}\left(\frac{n_b-(|S|-r)}{|S|-r+1}\right)$.

Proof. Part 1: Consider $(S,r) \in \mathcal{D}^G$. We proceed by induction on $|S|-r$. That $W(S,r) \le \frac{\sum_{j\in S} x_j}{|S|-r+1}$ for any $(S,r) \in \mathcal{D}^G$ with $|S|-r = 0$ follows from Lemma 4 part 2. Now suppose that $W(S,r) \le \frac{\sum_{j\in S} x_j}{|S|-r+1}$ for any $(S,r) \in \mathcal{D}^G$ with $|S|-r = k$, where $k \ge 0$. We need to prove that $W(S,r) \le \frac{\sum_{j\in S} x_j}{|S|-r+1}$ for any $(S,r) \in \mathcal{D}^G$ with $|S|-r = k+1$. Proof of this claim uses the following lemma.

Lemma 9. *Given* $n \ge 2$ *and* $r \in \{1,\ldots,n-1\}$, *for any* $(x_1,\ldots,x_n) \in \mathbb{R}^n_{++}$ *such that* $\sum_{i=1}^n x_i = c > 0$ *and* $x_i \le x_{i+1}$ *for any* $i \in \{1,\ldots,n-1\}$, *a partition of* $\{1,\ldots,n-1\}$ *into two subsets* A *and* B *exists such that* $\sum_{i\in A} x_i \le \frac{1}{n-r+1}c$ *and* $\sum_{i\in B} x_i \le \frac{n-r}{n-r+1}c$.

Proof. Fix $n \ge 2$, $r \in \{1,\ldots,n-1\}$, $\mathbf{x} \in \mathbb{R}^n_{++}$ such that $\sum_{i=1}^n x_i = c > 0$ and $x_i \le x_{i+1}$ $\forall i \in \{1,\ldots,n-1\}$. Let s be the largest integer such that $\sum_{i=1}^s x_i \le \frac{c}{n-r+1}$. Because $x_1 \le \frac{c}{n} \le \frac{c}{n-r+1}$, $s \ge 1$, and because $\sum_{i=1}^n x_i = c > \frac{c}{n-r+1}$, $s \le n-1$. Let $A = \{1,\ldots,s\}$ and $B = \{s+1,\ldots,n-1\}$. By construction $\sum_{i\in A} x_i \le \frac{1}{n-r+1}c$. It thus suffices to prove that $\sum_{i\in B} x_i \le \frac{n-r}{n-r+1}c$. If $B = \varnothing$, this is immediate. If $B \ne \varnothing$, suppose, towards a contradiction, that $\sum_{i\in B} x_i = \sum_{i=s+1}^{n-1} x_i > \frac{n-r}{n-r+1}c$. Because $B \ne \varnothing$, we have $s \le n-2$ and hence, by definition of s, $\sum_{i=1}^{s+1} x_i > \frac{1}{n-r+1}c$. Thus $\sum_{i=1}^{s+1} x_i + \sum_{i=s+1}^{n-1} x_i = x_{s+1} + \sum_{i=1}^{n-1} x_i > c = x_n + \sum_{i=1}^{n-1} x_i$ and hence $x_{s+1} > x_n$, which is a contradiction because $s+1 \le n-1 < n$ implies $x_{s+1} \le x_n$. ■

Because $(S,r) \in \mathcal{D}^G$ and $|S|-r = k+1 \ge 1$, we have $|S| \ge 2$ and $r \in \{1,\ldots,|S|-1\}$. Therefore, by Lemma 9, there exists a partition of S' into sets A and B such that $\sum_{j\in A} x_j \le \frac{\sum_{j\in S} x_j}{|S|-r+1}$ and $\sum_{j\in B} x_j \le \frac{(|S|-r)\sum_{j\in S} x_j}{|S|-r+1}$. Because $A \in 2^S$, (3) implies that $W(S,r) \le \max\{\sum_{j\in A} x_j, W((S\setminus A)', r-|A|)\}$. By construction of A and B, $\sum_{j\in A} x_j \le \frac{\sum_{j\in S} x_j}{|S|-r+1}$. Moreover, we have either (i) $|A| \ge r$, in which case $W((S\setminus A)', r-|A|) = 0$, or (ii) $|A| \le r-1$, in which case $((S\setminus A)', r-|A|) \in \mathcal{D}^G$ and $(S\setminus A)' = (B \cup \{\max S\})' = B$ and hence $W((S\setminus A)', r-|A|) \le \frac{\sum_{j\in B} x_j}{(|S|-|A|-1)-(r-|A|)+1} \le \frac{(|S|-r)\sum_{j\in S} x_j}{|S|-r+1}\frac{1}{|S|-r} = \frac{(\sum_{j\in S} x_j}{|S|-r+1}$, where the first inequality follows from the induction hypothesis.

Part 2: Let $\mathcal{D}^w = \{(n_a,n_b,r) \in \mathbb{N}^2_+ \times \mathbb{Z} | n_a+n_b-r \ge 0\}$ and for any $(n_a,n_b,r) \in \mathcal{D}^w$, let $w(n_a,n_b,r) = b\lceil \frac{n_b-(n_a+n_b-r)}{n_a+n_b-r+1}\rceil + a\left(\lceil \frac{-n_b}{n_a+n_b-r+1}\rceil + \lceil \frac{r}{n_a+n_b-r+1}\rceil\right)\mathbb{I}_{in}\left(\frac{n_b-(n_a+n_b-r)}{n_a+n_b-r+1}\right)$. We have the following: (i) for any $(n_a,n_b,r) \in \mathcal{D}^w$ with $n_a+n_b-r = 0$, $w(n_a,n_b,r) = bn_b + an_a$, (ii) for any $(n_a,n_b,r) \in \mathcal{D}^w$ with $n_a = 0$, because $\frac{r}{n_b-r+1} \in \mathbb{Z}$ implies that $\lceil \frac{-n_b}{n_b-r+1}\rceil = \lceil -\frac{n_b-r}{n_b-r+1} - \frac{r}{n_b-r+1}\rceil = -\frac{r}{n_b-r+1}$, $w(n_a,n_b,r) = b\lceil \frac{r}{n_b-r+1}\rceil$, (iii) for any $(n_a,n_b,r) \in \mathcal{D}^w$ with $n_b = 0$, because $\lceil -\frac{n_a-r}{n_a-r+1}\rceil = 0$, $w(n_a,n_b,r) = a\lceil \frac{r}{n_a-r+1}\rceil$, and (iv) for any $(n_a,n_b,r) \in \mathcal{D}^w$ with $r \le 0$, because $r \le 0$ implies that $\lceil -\frac{n_a-r}{n_a+n_b-r+1}\rceil = \lceil -\frac{n_b}{n_a+n_b-r+1}\rceil = \lceil -\frac{-r}{n_a+n_b-r+1}\rceil = 0$, $w(n_a,n_b,r) = 0$.

Suppose that $x_{(i)} = a$ for $i \in \{1, \dots, n_a\}$, $x_{(i)} = b$ for $i \in \{n_a + 1, \dots, n_a + n_b\}$, where $a, b > 0$, $n_a, n_b \in \{0, \dots, |S|\}$, $n_a + n_b = |S|$ and $b > 2|S|a$. We proceed by induction on $|S| - r$. That $W(S, r) = w(n_a, n_b, r)$ for any $(S, r) \in \mathcal{D}^G$ with $|S| - r = 0$ follows from property (i) of the w function because, from Lemma 4 part 2, $W(S, r) = \sum_{j \in S} x_j$ for any $(S, r) \in \mathcal{D}^G$ with $|S| - r = 0$.

Now suppose that $W(S, r) = w(n_a, n_b, r)$ for any $(S, r) \in \mathcal{D}^G$ with $|S| - r \leq k$, where $k \geq 0$. We need to prove that $W(S, r) = w(n_a, n_b, r)$ for any $(S, r) \in \mathcal{D}^G$ with $|S| - r = k + 1$. To see this, consider $(S, r) \in \mathcal{D}^G$ with $|S| - r = k + 1$, where $k \geq 0$. If $n_a = 0$ or $n_b = 0$, then $W(S, r) = w(n_a, n_b, r)$ by properties (ii) and (iii) of the w function because, by Lemma 4 part 10, $W(S, r) = x\lceil \frac{r}{|S|-r+1} \rceil$ for any $(S, r) \in \mathcal{D}^G$ with $x_i = x\ \forall i \in S$. Hence, suppose that $n_a \geq 1$ and $n_b \geq 1$.

We first claim that (3) given (S, r) admits solution T with $|\{i \in T | x_i = b\}| \leq n_b - 1$. Suppose that T solves (3) given (S, r) and that $|\{i \in T | x_i = b\}| = n_b$. By Lemma 4 part 1, $|T| \leq r \leq |S| - 1$ and hence $|S \setminus T| \geq 1$ so that, because $|\{i \in T | x_i = b\}| = n_b$, $x_{\max S \setminus T} = a$. Let $i_b \in T$ be such that $x_{i_b} = b$. Now consider $T_b = (T \setminus \{i_b\}) \cup \{\max S \setminus T\}$. By construction $\sum_{j \in T} x_j > \sum_{j \in T_b} x_j$. Moreover, $|T| = |T_b|$ and $(S \setminus T)' = (S \setminus T_b)'$ and thus $W((S \setminus T)', r - |T|) = W((S \setminus T_b)', r - |T_b|)$ and thus T_b solves (3) given (S, r). Hence, adding an additional constraint $|\{i \in T | x_i = b\}| \leq n_b - 1$ to the optimization problem in (3) does not change the value of the problem.

Second, by Lemma 4 part 5, the value of the objective function of the optimization problem in (3) evaluated at any two $T_1, T_2 \in 2^S$ such that $|\{i \in T_1 | x_i = a\}| = |\{i \in T_2 | x_i = a\}|$ and $|\{i \in T_1 | x_i = b\}| = |\{i \in T_2 | x_i = b\}|$ is the same. The optimization problem in (3) is thus equivalent to

$$\min_{\substack{m_a \in \{0, \dots, n_a\} \\ m_b \in \{0, \dots, n_b - 1\}}} \max \left\{ a m_a + b m_b, b\lceil \tfrac{n_b - m_b - (|S|-r)}{|S|-r} \rceil + a \left(\lceil \tfrac{m_b - (n_b - 1)}{|S|-r} \rceil + \lceil \tfrac{r - (m_a + m_b)}{|S|-r} \rceil \right) \mathbb{I}_{in}(\tfrac{n_b - m_b - (|S|-r)}{|S|-r}) \right\} \tag{5}$$

where the second term in the max operator is $W((S \setminus T)', r - |T|)$ which, by the induction hypothesis and property (iv) of the w function, equals w evaluated at $(n_a - m_a, n_b - m_b - 1, r - m_a - m_b)$.

For a given $m_a \in \{0, \dots, n_a\}$ and $m_b \in \{0, \dots, n_b - 1\}$, the first term in the max operator in (5) is at least $b m_b$ and, we claim, the second term is at least $b\lceil \frac{n_b - m_b - (|S|-r)}{|S|-r} \rceil$. This follows because $\frac{n_b - m_b - (|S|-r)}{|S|-r} = \frac{r - n_a - m_b}{|S|-r} \in \mathbb{Z}$ implies that $\lceil \frac{m_b - (n_b - 1)}{|S|-r} \rceil = \frac{m_b - n_b}{|S|-r} + \lceil \frac{1}{|S|-r} \rceil$ and that $\lceil \frac{r - (m_a + m_b)}{|S|-r} \rceil \geq \lceil \frac{r - n_a - m_b}{|S|-r} \rceil = \frac{r - n_a - m_b}{|S|-r}$ and hence $\lceil \frac{m_b - (n_b - 1)}{|S|-r} \rceil + \lceil \frac{r - (m_a + m_b)}{|S|-r} \rceil \geq -1 + \lceil \frac{1}{|S|-r} \rceil = 0$. Moreover, the first term is at most $b m_b + a2|S|$ and the second term is at most $b\lceil \frac{n_b - m_b - (|S|-r)}{|S|-r} \rceil + a2|S|$. The objective function in (5), for a given $m_b \in \{0, \dots, n_b - 1\}$, thus lies in the following interval

$$\left[b \max\{m_b, \lceil \tfrac{n_b - m_b - (|S|-r)}{|S|-r} \rceil\}, b \max\{m_b, \lceil \tfrac{n_b - m_b - (|S|-r)}{|S|-r} \rceil\} + a2|S| \right]. \tag{6}$$

Thus, because $\lceil \frac{n_b - m_b - (|S|-r)}{|S|-r} \rceil \in \mathbb{Z}$ for any $m_b \in \{0, \dots, n_b - 1\}$ and because $b > 2|S|a$, if

(m_a^*, m_b^*) solves (5), then m_b^* is a solution to

$$\min_{m_b \in \{0,\dots,n_b-1\}} b \max\left\{m_b, \lceil \tfrac{n_b - m_b - (|S|-r)}{|S|-r} \rceil\right\}. \tag{7}$$

We consider the following three cases.

Case 1: $n_b - (|S|-r) \leq 0$. Because $n_b - (|S|-r) \leq 0$, $m_b' = 0$ is the unique solution to (7) and the value of (7) is $b\max\{0,0\} = 0$. Hence $m_b^* = 0$ for any solution (m_a^*, m_b^*) to (5). Moreover, $n_b - (|S|-r) \leq 0$ implies that $0 \leq n_b - 1 < |S| - r$ and hence $\lceil -\frac{n_b-1}{|S|-r} \rceil = 0$. Thus $(m_a', 0)$ solves (5) if m_a' is the solution to

$$\min_{m_a \in \{0,\dots,n_a\}} a \max\left\{m_a, \lceil \tfrac{r-m_a}{|S|-r} \rceil\right\}. \tag{8}$$

The structure of (8) implies that it admits solution $m_a' = \lceil \frac{r}{|S|-r+1} \rceil$, where $m_a' \leq n_a$ is implied by $n_b - (|S|-r) = r - n_a \leq 0$. The value of (8) equals am_a' and hence the value of (5) is am_a'. What remains is to show that w evaluated at (n_a, n_b, r) such that $n_b - (|S|-r) \leq 0$ equals $a\lceil \frac{r}{|S|-r+1} \rceil$. This follows because $n_b - (|S|-r) \leq 0$ implies that $\lceil -\frac{(|S|-r)-n_b}{|S|-r+1} \rceil = 0$ and that $\lceil -\frac{n_b}{|S|-r+1} \rceil = 0$.

Case 2: $n_b - (|S|-r) > 0$ and $\frac{n_b-(|S|-r)}{|S|-r+1} \notin \mathbb{Z}$. In this case the structure of (7) implies that it admits solution $m_b' = \lceil \frac{n_b-(|S|-r)}{|S|-r+1} \rceil$, where $m_b' \leq n_b - 1$ is implied by $|S| - r \geq 1$. The value of (7) is bm_b'. If $\frac{n_b - m_b' - (|S|-r)}{|S|-r} \notin \mathbb{Z}$, then the objective function in (5) evaluated at $(0, m_b')$ equals bm_b'. If $\frac{n_b - m_b' - (|S|-r)}{|S|-r} \in \mathbb{Z}$, then $\lceil \frac{n_b - m_b' - (|S|-r)}{|S|-r} \rceil = m_b' - 1$ and hence the objective function in (5) evaluated at $(0, m_b')$ equals $\max\{bm_b', b(m_b'-1) + a(\lceil \frac{m_b' - (n_b-1)}{|S|-r} \rceil + \lceil \frac{r - m_b'}{|S|-r} \rceil)\} = bm_b'$, where the equality follows from $b > 2|S|a$. In either case, by the arguments leading to (6), $(0, m_b')$ solves (5) and its value is bm_b'. What remains is to show that w evaluated at (n_a, n_b, r) such that $n_b - (|S|-r) > 0$ and $\frac{n_b-(|S|-r)}{|S|-r+1} \notin \mathbb{Z}$ equals $b\lceil \frac{n_b-(|S|-r)}{|S|-r+1} \rceil$, which is immediate.

Case 3: $n_b - (|S|-r) > 0$ and $\frac{n_b-(|S|-r)}{|S|-r+1} \in \mathbb{Z}$. In this case the structure of (7) implies that $m_b' = \frac{n_b-(|S|-r)}{|S|-r+1}$ is the unique solution to (7), where $m_b' \leq n_b - 1$ is implied by $|S| - r \geq 1$. The value of (7) is $b\max\{m_b', m_b'\} = bm_b'$. Thus (m_a', m_b') solves (5) if m_a' is the solution to

$$\min_{m_a \in \{0,\dots,n_a\}} a \max\left\{m_a, \lceil \tfrac{m_b'-(n_b-1)}{|S|-r} \rceil + \lceil \tfrac{r-(m_a+m_b')}{|S|-r} \rceil\right\} = \min_{m_a \in \{0,\dots,n_a\}} a \max\left\{m_a, \lceil \tfrac{n_a - m_a}{|S|-r} \rceil\right\} \tag{9}$$

where the equality follows because we have $\frac{m_b'-n_b}{|S|-r} = -m_b' - 1$, and thus $\lceil \frac{m_b'-(n_b-1)}{|S|-r} \rceil = -m_b' - 1 + \lceil \frac{1}{|S|-r} \rceil = -m_b'$, and $\frac{r-m_b'}{|S|-r} = m_b' + \frac{n_a}{|S|-r}$, and thus $\lceil \frac{r-(m_a+m_b')}{|S|-r} \rceil = m_b' + \lceil \frac{n_a-m_a}{|S|-r} \rceil$. The structure of (9) implies that it admits solution $m_a' = \lceil \frac{n_a}{|S|-r+1} \rceil$ and its value is am_a'. Hence (m_a', m_b') solves (5) and its value is $bm_b' + am_a'$. What remains is to show that w evaluated at (n_a, n_b, r) such that $n_b - (|S|-r) > 0$ and $\frac{n_b-(|S|-r)}{|S|-r+1} \in \mathbb{Z}$ equals $b\lceil \frac{n_b-(|S|-r)}{|S|-r+1} \rceil + a\lceil \frac{n_a}{|S|-r+1} \rceil$. This follows because $\frac{n_b-(|S|-r)}{|S|-r+1} \in \mathbb{Z}$ and because $\lceil \frac{n_a-|S|}{|S|-r+1} \rceil =$

$\lceil \frac{-n_b}{|S|-r+1} \rceil = \lceil -m'_b - \frac{|S|-r}{|S|-r+1} \rceil = -m'_b$ and $\lceil \frac{r}{|S|+r-1} \rceil = \lceil \frac{n_a}{|S|-r+1} + m'_b \rceil = \lceil \frac{n_a}{|S|-r+1} \rceil + m'_b$. ■

Consider the profile $\mathbf{x}(\varepsilon, n_b) = (x(\varepsilon, n_b)_i)_{i \in S}$ such that $x(\varepsilon, n_b)_{(i)} = \varepsilon$ for $i \in \{1, \ldots, |S| - n_b\}$, $x(\varepsilon, n_b)_{(i)} = \frac{c - \varepsilon(|S| - n_b)}{n_b}$ for $i \in \{|S| - n_b + 1, \ldots, |S|\}$, where $c > 0$ and $n_b = k(|S| - r + 1)$ for some $k \in \{1, \ldots, \lfloor \frac{|S|}{|S|-r+1} \rfloor\}$, where $\{1, \ldots, \lfloor \frac{|S|}{|S|-r+1} \rfloor\} \neq \varnothing$ because $\frac{|S|}{|S|-r+1} \geq 1$. Then there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon})$, $\mathbf{x}(\varepsilon, n_b) \in \mathbb{R}^{|S|}_{++}$ and $\sum_{i \in S} x(\varepsilon, n_b)_i = c$. Moreover, by Lemma 8 part 2, given $\mathbf{x}(\varepsilon, n_b)$, if $(S, r) \in \mathcal{D}^G$ and $r < |S|$, we have $W(S, r) = \frac{c - \varepsilon(|S| - n_b)}{n_b} \lceil \frac{n_b - (|S| - r)}{|S| - r + 1} \rceil = \frac{c - \varepsilon(|S| - n_b)}{k(|S| - r + 1)} \lceil k - \frac{|S| - r}{|S| - r + 1} \rceil = \frac{c - \varepsilon(|S| - n_b)}{|S| - r + 1}$. At the same time Lemma 8 part 1 implies that $W(S, r) \leq \frac{c}{|S| - r + 1}$ for any profile $(x_i)_{i \in S} \in \mathbb{R}^{|S|}_{++}$ such that $\sum_{i \in S} x_i = c$. The $\mathbf{x}(\varepsilon, n_b)$ thus comes arbitrarily close to the upper bound on $W(S, r)$ as $\varepsilon \to 0$.

Proposition 12. *Consider a subgame starting with state $(S, r) \in \mathcal{D}^G$. Suppose $y \notin \mathcal{L}$ and $\delta > \bar{\delta}$.*

1. *An equilibrium exists. If multiple equilibria exist, then*
   (a) *the leader's payoff is constant across equilibria, and*
   (b) *for all $i \in N \setminus S$ the payoff of member i is constant across equilibria.*
2. *If the policy passes in an equilibrium, then*
   (a) *it passes in $r + 1$ rounds,*
   (b) *the equilibrium consists of two phases (possibly empty), a temptation phase followed by an exploitation phase,*
   (c) *the set of members included in the temptation phase solves (4), and*
   (d) *if $r < |S|$ and $x_{(r)} < x_i$ for some member $i \in S$, then member i is not included in the temptation phase,*
   (e) *the limit of the leader's payoff is $y - \Pi(S, r, y)$ as $\delta \to 1$.*
3. *If $y > W(S, r)$, then the policy passes in all equilibria.*
4. *If $y < W(S, r)$, then the policy does not pass in any equilibrium.*
5. *In an equilibrium, if member $i \in S$ is approached in state (S, r), then he accepts a transfer if and only if it is weakly greater than a cutoff. In state (S, r), if i is dispensable his cutoff is $x_i \delta^r (1 - \delta)$, if i is indispensable his cutoff is $x_i \delta^r$ and if i is inconsequential his cutoff is 0.*

Proof. We prove Proposition 12 by induction on the size of $|S|$. Before the induction argument, we deal with several preliminaries. First, we introduce two functions l and c that define a space of strategies Σ and during the proof we (inductively) specify the l and c functions such that a profile σ constitutes an equilibrium if and only if $\sigma \in \Sigma$. For any $(S, r) \in \mathcal{D}^W$, let $l(S, r) \subseteq \{\text{initiate a vote, stop}\} \cup (S \times \mathbb{R}_+)$ and for any state $(S, r) \in \mathcal{D}^W$ and any $i \in S$, let $c(S, r, i) \in \mathbb{R}_+$. Let $H = \cup_{r \in \mathbb{Z}, r \leq n} H_r$, where H_r is the set of possible histories in $\Gamma(N, r)$. At any $h \in H$, either the leader moves or $h = (h_l, (i, t))$, that is, a member i responds to the leader's offer t made at history h_l. In the former case, let (S^h, r^h) be the state that corresponds to h. In the latter case, let (S^h, r^h) be the state that corresponds to h_l. Let $\tilde{\sigma}_l$ be a strategy of the leader that satisfies $\tilde{\sigma}_l(h) \in l(S^h, r^h)$

for any $h \in H$ at which the leader moves and let Σ_l be the space of all $\tilde{\sigma}_l$ strategies. Let $\tilde{\sigma}_i$ be the strategy of member $i \in N$ such that, for each history $h \in H$ at which the leader approaches i with an offer t, $\tilde{\sigma}_i(h) =$ accept if and only if $t \geq c(S^h, r^h, i)$. Let $\tilde{\sigma} = (\tilde{\sigma}_l, (\tilde{\sigma}_i)_{i \in N})$ be a profile of strategies constructed from the l and c functions and $\Sigma = \Sigma_l \times (\times_{i \in N}\{\tilde{\sigma}_i\})$ be the space of all $\tilde{\sigma}$ profiles.

States $(S,r) \in \mathcal{D}^W$ with $r \leq 0$. For any $(S,r) \in \mathcal{D}^W$ with $r \leq 0$ and any $i \in S$, set $l(S,r) = \{\text{initiate a vote}\}$ and $c(S,r,i) = 0$. Fix $(S,r) \in \mathcal{D}^W$ with $r \leq 0$. $\Gamma(S,r)$ admits unique equilibrium in which the principal initiates a vote at any history in which she moves and any member accepts any transfer at any history in which he moves. Thus, a profile σ constitutes an equilibrium in $\Gamma(S,r)$ if and only if $\sigma \in \Sigma$ and the equilibrium payoff from (S,r) is y for the principal and $-x_i$ for any member $i \in N$.

States $(S,r) \in \mathcal{D}^W$ with $r > |S|$. For any $(S,r) \in \mathcal{D}^W$ with $r > |S|$ and any $i \in S$, set $l(S,r) = \{\text{initiate a vote}, \text{stop}\} \cup (S \times \{0\})$ and $c(S,r,i) = 0$. Fix $(S,r) \in \mathcal{D}^W$ with $r > |S|$. In any equilibrium of $\Gamma(S,r)$ the policy does not pass, hence any approached member accepts any offered transfer and thus the leader never offers strictly positive transfer to any member. Therefore, a profile σ constitutes an equilibrium in $\Gamma(S,r)$ only if $\sigma \in \Sigma$. Moreover, any $\tilde{\sigma} \in \Sigma$ constitutes an equilibrium in $\Gamma(S,r)$ and thus the equilibrium payoff from (S,r) is 0 for the leader and for any member $i \in N$.

Initial induction step. We now prove Proposition 12 for any $(S,r) \in \mathcal{D}^G$ with $|S| = 1$. Notice that $(S,r) \in \mathcal{D}^G$ and $|S| = 1$ implies $r = 1$ and thus $W(S,r) = \Pi(S,r,y) = x_{(1)}$. For any $(S,r) \in \mathcal{D}^G$ with $|S| = 1$ and any $i \in S$, set $l(S,r) = \{(i, x_i)\}$ if $y > x_i$, $l(S,r) = \{\text{initiate a vote}, \text{stop}, (i,0)\}$ if $y < x_i$ and $c(S,r,i) = \delta x_i$. Fix $(S,r) \in \mathcal{D}^G$ with $|S| = 1$ and let $S = \{i\}$. In any equilibrium of $\Gamma(S,r)$, if the leader in (S,r) approaches i with transfer t, then i's payoff from rejection is 0, because the game moves to state $(\varnothing, 1)$, and i's payoff from accepting is $t - \delta x_i$, because the game moves to state $(\varnothing, 0)$. Hence, if approached in (S,r), i accepts t if and only if $t \geq \delta x_i$. Note that i is indispensable in (S,r) because $W(\varnothing, 0) = 0 < y < W(\varnothing, 1) = \infty$. For the leader, thus, the payoff in (S,r) from initiating a vote or stoping is 0 and the payoff from approaching member i with an offer t is 0 if $t \in [0, \delta x_i)$ and is $\delta y - t$ if $t \geq \delta x_i$. Hence, the leader in (S,r) approaches i with offer δx_i if $y > x_i$ and either initiates a vote or stops or approaches i with offer 0 if $y < x_i$. Therefore, a profile σ constitutes an equilibrium in $\Gamma(S,r)$ only if $\sigma \in \Sigma$. Moreover, any $\tilde{\sigma} \in \Sigma$ constitutes an equilibrium in $\Gamma(S,r)$ and thus the equilibrium payoff from (S,r) is $\delta(y - x_i)$ for the leader, 0 for member i and $-x_j$ for any member $j \in N \setminus \{i\}$ if $y > x_i$ and is 0 for the leader and for any member $j \in N$ if $y < x_i$. Because $y \notin \mathcal{L}$ and $x_i \in \mathcal{L}$, we have $y \neq x_i$ and this concludes the proof of Proposition 12 for all states $(S,r) \in \mathcal{D}^G$ with $|S| = 1$.

Induction step from k to $k+1$. Assume Proposition 12 holds for all states $(S,r) \in \mathcal{D}^G$ with $|S| \leq k$, where $k \geq 1$. We now prove Proposition 12 for any $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$. Fix $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$.

First, we prove part 5. In any equilibrium of $\Gamma(S,r)$, suppose the leader in (S,r) approaches $i \in S$ with transfer t. Let v_a be i's payoff from accepting and v_r be i's payoff from rejecting. There are three cases to consider.

Case 1: $r = 1$. If $r = 1$, we have $v_a = t - \delta x_i$, because the game proceeds to state $(S \setminus \{i\}, r-1)$, and, by the induction hypothesis, $v_r = 0$ if $y < W(S \setminus \{i\}, r)$ and $v_r = -\delta^2 x_i$ if $y > W(S \setminus \{i\}, r)$. In the former case, i accepts the leader's offer t if and only if $t \geq \delta x_i$ and in the latter case i accepts the leader's offer t if and only if $t \geq \delta x_i(1-\delta)$. Note that, because $W(S \setminus \{i\}, r-1) = 0$ when $r = 1$, in the former case i is indispensable in (S, r) and in the latter case i is dispensable in (S, r).

Case 2: $r = |S|$. If $r = |S|$, we have $v_r = 0$, because the game proceeds to state $(S \setminus \{i\}, r)$, and, by the induction hypothesis, $v_a = t$ if $y < W(S \setminus \{i\}, r-1)$ and $v_a = t - \delta^r x_i$ if $y > W(S \setminus \{i\}, r-1)$. In the former case i accepts the leader's offer t if and only if $t \geq 0$ and in the latter case i accepts the leader's offer t if and only if $t \geq \delta^r x_i$. Note that, because $W(S \setminus \{i\}, r) = \infty$ when $r = |S|$, in the former case i is inconsequential in (S, r) and in the latter case i is indispensable in (S, r).

Case 3: $r \in \{2, \ldots, |S|-1\}$. Because $y \notin \mathcal{L}$, because $W(S \setminus \{i\}, r-1), W(S \setminus \{i\}, r) \in \mathcal{L}$ by Lemma 4 part 3 and because $W(S \setminus \{i\}, r-1) \leq W(S \setminus \{i\}, r)$ by Lemma 4 part 4, there are three cases to consider: in (S, r), i is either inconsequential, when $y < W(S \setminus \{i\}, r-1)$, or indispensable, when $y \in (W(S \setminus \{i\}, r-1), W(S \setminus \{i\}, r))$, or dispensable, when $y > W(S \setminus \{i\}, r)$. By the induction hypothesis, if i is inconsequential we have $v_a = t$ and $v_r = 0$ and i accepts the leader's offer t if and only if $t \geq 0$, if i is indispensable we have $v_a = t - \delta^r x_i$ and $v_r = 0$ and i accepts the leader's offer t if and only if $t \geq \delta^r x_i$, and if i is dispensable we have $v_a = t - \delta^r x_i$ and $v_r = -\delta^{r+1} x_i$ and i accepts the leader's offer t if and only if $t \geq \delta^r x_i (1 - \delta)$.

For any $i \in S$, set $c(S, r, i) = 0$ if i is inconsequential in (S, r), set $c(S, r, i) = \delta^r x_i$ if i is indispensable in (S, r) and set $c(S, r, i) = \delta^r x_i (1 - \delta)$ if i is dispensable in (S, r).

We now prove that $\Gamma(S, r)$ admits an equilibrium and that the leader's payoff is constant across equilibria, parts 1 and 1a. By construction, a profile σ constitutes an equilibrium in any proper subgame of $\Gamma(S, r)$ if and only if $\sigma \in \Sigma$. It thus suffices to prove that, for any $\tilde{\sigma} \in \Sigma$, the leader has an optimal action at the initial history of $\Gamma(S, r)$ when her payoff in proper subgames of $\Gamma(S, r)$ is determined by $\tilde{\sigma}$, and that the payoff the optimal action provides to the leader is independent of $\tilde{\sigma}$. That the payoff the optimal action provides to the leader is independent of $\tilde{\sigma}$ follows from the induction hypothesis; the leader's payoff from states $(S \setminus \{i\}, r)$ and $(S \setminus \{i\}, r-1)$ is, for any $i \in S$, constant across equilibria and the payoff from initiating a vote or stopping is 0.

To shows that the leader has an optimal action at the initial history of $\Gamma(S, r)$, fix $\tilde{\sigma} \in \Sigma$ and, $\forall i \in S$, let a_i be the leader's equilibrium payoff from $(S \setminus \{i\}, r-1)$ and let r_i be the leader's equilibrium payoff from $(S \setminus \{i\}, r)$. Let $A = \{\text{initiate a vote}, \text{stop}\} \cup (S \times \mathbb{R}_+)$ be the leader's action space at the initial history of $\Gamma(S, r)$ and let $v : A \to \mathbb{R}$ be the leader's payoff function at the initial history of $\Gamma(S, r)$. Clearly, $v(\text{initiate a vote}) = v(\text{stop}) = 0$. For any $(i, t) \in S \times \mathbb{R}_+$, we have

$$v(i, t) = \begin{cases} \delta r_i & \text{if } t < c(S, r, i) \\ \delta a_i - t & \text{if } t \geq c(S, r, i) \end{cases}. \tag{10}$$

The leader's payoff maximization problem reads $\max_{a \in A} v(a)$. For any $i \in S$, $\max_{t \in \mathbb{R}_+} v(i, t)$ has a solution and we can set $v_i = \max_{t \in \mathbb{R}_+} v(i, t)$. Because $\max_{a \in A} v(a)$ is equivalent to $\max \{0, \max_{i \in S} \{v_i\}\}$ and because the latter problem is finite and hence admits a solution, the former problem admits a solution as well. Set $l(S, r) = \arg\max_{a \in A} v(a)$. Part 1b follows from parts 2a, 3 and 4 we prove below.

We now prove part 2a. By construction, σ constitutes an equilibrium in $\Gamma(S, r)$ if and only if $\sigma \in \Sigma$. Fix $\tilde{\sigma} \in \Sigma$ in which the policy passes and suppose, towards a contradiction, that on the equilibrium path the leader approaches $r+1$ or more members. If the member i

approached in (S, r) accepts the leader's offer then the game proceeds into state $(S\setminus\{i\}, r-1)$ in which, by the induction hypothesis, the leader on the equilibrium path approaches $r-1$ members. Hence, it must be the case that the leader in (S, r) approaches $i_0 \in S$ with an offer $t_0 < c(S, r, i_0)$, the game proceeds to $(S\setminus\{i_0\}, r)$ and starting from $(S\setminus\{i_0\}, r)$ the leader's equilibrium sequence of actions is $(i_a, t_a)_{a=1}^r$ along which all approached members accept. For $a \in \{1, \ldots, r\}$, let $S_a = S \setminus \cup_{c=0}^{a-1}\{i_c\}$. Then the equilibrium sequence of states is $(S_a, r+1-a)_{a=1}^r$ and, $\forall a \in \{1, \ldots, r\}$, i_a is offered t_a in state $(S_a, r+1-a)$ and hence, because i_a accepts t_a, we have $t_a = c(S_a, r+1-a, i_a)$. Because the policy passes in $\tilde{\sigma}$, all agents in $(i_a)_{a=1}^r$, when approached, are either indispensable or dispensable. Let $T = \{i_a | a \in \{1, \ldots, r\}, y < W(S_a \setminus \{i_a\}, r+1-a)\}$ be the set of indispensable agents approached and let $E = \{i_a | a \in \{1, \ldots, r\}, y > W(S_a \setminus \{i_a\}, r+1-a)\}$ be the set of dispensable agents approached. By construction, $\forall a \in \{1, \ldots, r\}$, we have $c(S_a, r+1-a, i_a) = \delta^{r+1-a}x_{i_a}$ if $i_a \in T$ and $c(S_a, r+1-a, i_a) = \delta^{r+1-a}x_{i_a}(1-\delta)$ if $i_a \in E$. The leader's equilibrium payoff from (S, r) under $\tilde{\sigma}$ is thus

$$\delta^{r+1}y - \sum_{a\in\{1,\ldots,r\}, i_a \in T} \delta^a \delta^{r+1-a} x_{i_a} - \sum_{a\in\{1,\ldots,r\}, i_a\in E} \delta^a\delta^{r+1-a}x_{i_a}(1-\delta) = \delta^{r+1}\left(y - \sum_{i\in T} x_i - \sum_{i\in E} x_i(1-\delta)\right). \tag{11}$$

Because $y \notin \mathcal{L}$, we have $y \neq \sum_{i\in T} x_i$ and hence, because $\tilde{\sigma}$ constitutes an equilibrium, $y > \sum_{i\in T} x_i$. Because $\delta > \bar{\delta} \geq \bar{\delta}_a$, we thus have $y - \sum_{i\in T} x_i - \sum_{i\in E} x_i(1-\delta) > 0$.

We now construct $\tilde{\sigma}'$ that constitutes a profitable deviation for the leader. For $a \in \{1, \ldots, r\}$, let $S_a^\circ = S \setminus \cup_{c=1}^{a-1}\{i_c\}$ and note that $S_a^\circ \setminus \{i_0\} = S_a$. Let $\tilde{\sigma}'$ be identical to $\tilde{\sigma}$ except that the leader, $\forall a \in \{1, \ldots, r\}$, approaches member i_a with an offer t_a at any history that corresponds to state $(S_a^\circ, r+1-a)$. We now argue that, $\forall a \in \{1, \ldots, r\}$, member i_a offered t_a in state $(S_a^\circ, r+1-a)$ accepts. For any $i_a \in T$ this is immediate because we have $c(S_a, r+1-a, i_a) = \delta^{r+1-a}x_{i_a} \geq c(S_a^\circ, r+1-a, i_a)$. For any $i_a \in E$, we have $y > W(S_a \setminus \{i_a\}, r+1-a) = W(S_a^\circ \setminus \{i_0, i_a\}, r+1-a) \geq W(S_a^\circ \setminus \{i_a\}, r+1-a)$, where the weak inequality follows by Lemma 4 part 5, and hence $c(S_a^\circ, r+1-a, i_a) = \delta^{r+1-a}x_{i_a}(1-\delta) = c(S_a, r+1-a, i_a)$. Because, $\forall a \in \{1, \ldots, r\}$, member i_a offered t_a in state $(S_a^\circ, r+1-a)$ accepts, the leader's payoff from (S, r) under $\tilde{\sigma}'$ is $\delta^r(y - \sum_{i\in T} x_i - \sum_{i\in E} x_i) > \delta^{r+1}(y - \sum_{i\in T} x_i - \sum_{i\in E} x_i)$, which establishes the desired contradiction.

To prove part 2b, it suffices to prove that given any $\tilde{\sigma} \in \Sigma$ in which the policy passes, if there is a dispensable member in (S, r), then the leader at the initial history of $\Gamma(S, r)$ approaches a dispensable member i with an offer that i accepts and that there is a dispensable member in the state the game proceeds to, in $(S \setminus \{i\}, r-1)$. The see the latter claim, if $i \in S$ is dispensable in (S, r) we have $y > W(S \setminus \{i\}, r)$, by Lemma 4 part 6 we have, $\forall j \in S \setminus \{i\}$, $W(S \setminus \{i\}, r) \geq W(S \setminus \{i, j\}, r-1)$, and thus any $j \in S \setminus \{i\}$ is dispensable in $(S\setminus\{i\}, r-1)$. To see the former claim, fix $\tilde{\sigma} \in \Sigma$ in which the policy passes. Let $E = \{i \in S | y > W(S \setminus \{i\}, r)\}$, $T = \{i \in S | y \in (W(S \setminus \{i\}, r-1), W(S \setminus \{i\}, r))\}$ and $Z = \{i \in S | y < W(S \setminus \{i\}, r-1)\}$ be, respectively, the set of dispensable, indispensable and inconsequential members in (S, r). The leader in equilibrium does not approach $i \in Z$ at the initial history of $\Gamma(S, r)$ because the policy passes in $\tilde{\sigma}$. Suppose, towards a contradiction, that the leader in equilibrium at the initial history of $\Gamma(S, r)$ approaches $i_0 \in T$ and $E \neq \varnothing$. By part 2a, the leader in equilibrium approaches i_0 with offer t_0 that

i_0 accepts, that is, with $t_0 = c(S, r, i_0) = \delta^r x_{i_0}$, where the second equality follows from $i_0 \in T$, and hence the leader's equilibrium payoff from (S, r) is at most $\delta^r(y - x_{i_0})$.

Let $(i_a)_{a=1}^r$ be a sequence of members and, $\forall a \in \{1, \dots, r\}$, let $S_a = S \setminus \cup_{c=1}^{a-1}\{i_c\}$ and $t_a = \delta^{r+1-a} x_{i_a}(1-\delta)$. Suppose that the sequence of members $(i_a)_{a=1}^r$ is such that $i_1 \in E$ and $i_a \in S_a$ for any $a \in \{2, \dots, r\}$. Consider $\tilde{\sigma}'$ identical to $\tilde{\sigma}$ except that the leader, $\forall a \in \{1, \dots, r\}$, approaches member i_a with an offer t_a at any history that corresponds to state $(S_a, r+1-a)$. We now argue that, $\forall a \in \{1, \dots, r\}$, member i_a offered t_a in state $(S_a, r+1-a)$ accepts. To see this, it suffices to argue that, $\forall a \in \{1, \dots, r\}$, i_a is dispensable in state $(S_a, r+1-a)$. By construction, $i_1 \in E$ and hence i_1 is dispensable in $(S, r) = (S_1, r)$. Note that because i_1 is dispensable in (S, r), we have $y > W(S \setminus \{i_1\}, r)$. By Lemma 4 part 6, we have, $\forall a \in \{2, \dots, r\}$, $W(S \setminus \{i_1\}, r) \geq W(S \setminus \cup_{c=1}^{a}\{i_c\}, r+1-a) = W(S_a \setminus \{i_a\}, r+1-a)$. Therefore, $\forall a \in \{2, \dots, r\}$, i_a is dispensable in $(S_a, r+1-a)$. Because the leader approaches all members in the $(i_a)_{a=1}^r$ sequence with offers the members accept, her payoff from $\tilde{\sigma}'$ is $\delta^r y - \sum_{a \in \{1,\dots,r\}} \delta^{a-1}\delta^{r+1-a} x_{i_a}(1-\delta) = \delta^r(y - \sum_{a\in\{1,\dots,r\}} x_{i_a}(1-\delta))$. Because $\delta > \bar{\delta} \geq \bar{\delta}_b$, we have $x_{i_0} \geq x_1 > n x_n(1-\delta) \geq \sum_{a\in\{1,\dots,r\}} x_{i_a}(1-\delta)$ and thus $y - x_{i_0} < y - \sum_{a\in\{1,\dots,r\}} x_{i_a}(1-\delta)$, which establishes the desired contradiction.

We now prove part 2c. Fix $\tilde{\sigma} \in \Sigma$ in which the policy passes. Let $(i_a)_{a=1}^r$ be the equilibrium sequence of approached members. By part 2a, this sequence consists of r members. For $a \in \{1, \dots, r\}$, let $S_a = S \setminus \cup_{c=1}^{a-1}\{i_c\}$ and $t_a = c(S_a, r+1-a, i_a)$. On the equilibrium path, $\forall a \in \{1, \dots, r\}$, the leader approaches i_a with an offer t_a in state $(S_a, r+1-a)$ and i_a accepts t_a. Let $T = \{i_a | a \in \{1, \dots, r\}, y < W(S_a \setminus \{i_a\}, r+1-a)\}$ be the set indispensable agents approached and let $E = \{i_a | a \in \{1, \dots, r\}, y > W(S_a \setminus \{i_a\}, r+1-a)\}$ be the set of dispensable agents approached. By part 5, $t_a = \delta^{r+1-a} x_{i_a}$ for any $i_a \in T$ and $t_a = \delta^{r+1-a} x_{i_a}(1-\delta)$ for any $i_a \in E$. The leader's equilibrium payoff from $\tilde{\sigma}$ is thus $\delta^r(y - \sum_{i\in T} x_i - \sum_{i\in E} x_i(1-\delta))$.

When $r = |S|$, we have $T = S$ because, $\forall a \in \{1, \dots, r\}$, $W(S_a \setminus \{i_a\}, r+1-a) = \infty$. By Lemma 5 part 2, S is a solution to (4). Hence, suppose $r < |S|$. Clearly, $T \in 2^S$ and we argue that $y > W((S \setminus T)', r - |T|)$. We have either $|T| = r$ or $|T| < r$. In the former case, $W((S \setminus T)', r - |T|) = 0$. In the latter case, for the first dispensable member approached, $i_{|T|+1}$, we have $y > W(S_{|T|+1} \setminus \{i_{|T|+1}\}, r - |T|)$. By part 2b, all indispensable members are approached before any dispensable member is approached and hence $S_{|T|+1} = S \setminus T$. Because $i_{|T|+1}$ is approached in state $(S_{|T|+1}, r-|T|) = (S \setminus T, r-|T|)$, we have $i_{|T|+1} \leq \max S \setminus T$ and hence, by Lemma 4 part 5, $W((S \setminus T) \setminus \{i_{|T|+1}\}, r - |T|) \geq W((S \setminus T) \setminus \{\max S \setminus T\}, r - |T|)$. Thus $y > W((S \setminus T)', r - |T|)$. Because $T \in 2^S$ and $y > W((S \setminus T)', r - |T|)$, it suffices to prove that $T_o \in 2^S$ such that $\sum_{j\in T} x_j > \sum_{j\in T_o} x_j$ and $y > W((S \setminus T_o)', r - |T_o|))$ does not exist.

Suppose, towards a contradiction, that $T_o \in 2^S$ such that $\sum_{j\in T} x_j > \sum_{j\in T_o} x_j$ and $y > W((S \setminus T_o)', r - |T_o|))$ exists. If $|T_o| > r$, then any $T_b \subseteq T_o$ such that $|T_b| = r$ satisfies $\sum_{j\in T_o} x_j > \sum_{j\in T_b} x_j$ and $W((S \setminus T_o)', r - |T_o|)) = W((S \setminus T_b)', r - |T_b|))$ and hence it is without loss of generality to assume that $|T_o| \leq r$. Let $(i_a^\circ)_{a=1}^r$ be a sequence of members and, $\forall a \in \{1, \dots, r\}$, let $S_a^\circ = S \setminus \cup_{c=1}^{a-1}\{i_c^\circ\}$. Suppose that the sequence of members $(i_a^\circ)_{a=1}^r$ is such that $i_a^\circ \in T_o$ $\forall a \in \{1, \dots, |T_o|\}$, $i_{|T_o|+1}^\circ = \max S \setminus T_o$, and $i_a^\circ \in S_a^\circ$ $\forall a \in \{|T_o|+2, \dots, r\}$. Let $E_o = \{|T_o|+1, \dots, r\}$. Let $t_a^\circ = \delta^{r+1-a} x_{i_a^\circ}$ for any $i_a^\circ \in T_o$ and $t_a^\circ = \delta^{r+1-a} x_{i_a^\circ}(1-\delta)$ for any $i_a^\circ \in E_o$. Consider $\tilde{\sigma}'$ identical to $\tilde{\sigma}$ except that the leader, $\forall a \in \{1, \dots, r\}$, approaches member i_a° with an offer t_a° at any history that corresponds to state $(S_a^\circ, r+1-a)$. We now argue that, $\forall a \in \{1,\dots,r\}$, member i_a° offered t_a° in state $(S_a^\circ, r+1-a)$ accepts. To see this, for any $i_a^\circ \in T_o$ we have $t_a^\circ = \delta^{r+1-a} x_{i_a^\circ}$ and hence member i_a° offered t_a° in state $(S_a^\circ, r+1-a)$ accepts. To see that any $i_a^\circ \in E_o$ accepts t_a° in $(S_a^\circ, r+1-a)$, it suffices to argue that any $i_a^\circ \in E_o$ is dispensable in $(S_a^\circ, r+1-a)$. Member $i_{|T_o|+1}^\circ = \max S \setminus T_o$ is approached in $(S_{|T_o|+1}^\circ, r-|T_o|) = (S \setminus T_o, r-|T_o|)$ and we have $y > W((S \setminus T_o)', r-|T_o|)$. Thus $i_{|T_o|+1}^\circ$ is dispensable in $(S_{|T_o|+1}^\circ, r-|T_o|)$. Because $i_{|T_o|+1}^\circ$ is dispensable in $(S_{|T_o|+1}^\circ, r-|T_o|)$, we have $y > W(S \setminus \cup_{c=1}^{|T_o|+1}\{i_c^\circ\}, r+1-(|T_o|+1))$. By Lemma 4 part 6, we have, $\forall a \in \{|T_o|+2,\dots,r\}$, $W(S \setminus \cup_{c=1}^{|T_o|+1}\{i_c^\circ\}, r+1-(|T_o|+1)) \geq W(S \setminus \cup_{c=1}^{a}\{i_c^\circ\}, r+1-a)$. Thus, $\forall a \in \{|T_o|+2,\dots,r\}$, $y > W(S \setminus \cup_{c=1}^{a}\{i_c^\circ\}, r+1-a)$ and thus i_a° is dispensable in $(S_a^\circ, r+1-a)$. Because the leader approaches all members in the $(i_a^\circ)_{a=1}^r$ sequence with offers the members accept, her payoff from $\tilde{\sigma}'$ is $\delta^r(y - \sum_{i\in T_o} x_i - \sum_{i \in E_o} x_i(1-\delta))$. Because $\delta > \bar{\delta} \geq \bar{\delta}_c$, we have $\sum_{i\in T_o} x_i + \sum_{i\in E_o} x_i(1-\delta) \leq \sum_{i \in T_o} x_i + n x_n(1-\delta) < \sum_{i\in T} x_i \leq \sum_{i\in T} x_i + \sum_{i\in E} x_i(1-\delta)$ and thus $y - \sum_{i\in T} x_i - \sum_{i\in E} x_i(1-\delta) < y - \sum_{i\in T_o} x_i - \sum_{i\in E_o} x_i(1-\delta)$, which establishes the desired contradiction. Part 2d now follows from part 2c and Lemma 5 part 9.

We now prove part 2e. Fix $\tilde{\sigma} \in \Sigma$ in which the policy passes. Let $(i_a)_{a=1}^r$ be the equilibrium sequence of approached members. By part 2a, this sequence consists of r members. For $a \in \{1,\dots,r\}$, let $S_a = S \setminus \cup_{c=1}^{a-1}\{i_c\}$ and $t_a = c(S_a, r+1-a, i_a)$. On the equilibrium path, $\forall a \in \{1,\dots,r\}$, the leader approaches i_a with an offer t_a in state $(S_a, r+1-a)$ and i_a accepts t_a. Let $T = \{i_a | a \in \{1,\dots,r\}, y < W(S_a \setminus \{i_a\}, r+1-a)\}$ be the set indispensable agents approached and let $E = \{i_a | a \in \{1,\dots,r\}, y > W(S_a \setminus \{i_a\}, r+1-a)\}$ be the set of dispensable agents approached. By part 5, $t_a = \delta^{r+1-a} x_{i_a}$ for any $i_a \in T$ and $t_a = \delta^{r+1-a} x_{i_a}(1-\delta)$ for any $i_a \in E$. The leader's equilibrium payoff from $\tilde{\sigma}$ is thus $\delta^r(y - \sum_{i\in T} x_i - \sum_{i\in E} x_i(1-\delta))$. By part 2c, T is a solution to (4) and hence $\sum_{j\in T} x_j = \Pi(S,r,y)$. We thus have $l(\delta) \leq \delta^r(y - \sum_{i\in T} x_i - \sum_{i\in E} x_i(1-\delta)) \leq u(\delta)$, where

$$\begin{aligned} l(\delta) &= \delta^r(y - \Pi(S,r,y) - (1-\delta) r x_n) \\ u(\delta) &= y - \Pi(S,r,y). \end{aligned} \tag{12}$$

The result now follows by the squeeze lemma because $\lim_{\delta\to 1} l(\delta) = \lim_{\delta \to 1} u(\delta) = y - \Pi(S,r,y)$

We now prove part 3. Suppose, towards a contradiction, that $y > W(S,r)$ and the policy does not pass in some $\tilde{\sigma} \in \Sigma$. Fix $\tilde{\sigma} \in \Sigma$ in which the policy does not pass. The leader's payoff from $\tilde{\sigma}$ is thus 0. If $r = |S|$, set $T = S$. Because $W(S,r) = \sum_{i\in S} x_i$ when $r = |S|$, $y > W(S,r)$ implies $y > \sum_{i\in S} x_i$. If $r < |S|$, let $T \in 2^S$ be a solution to (3). Thus $W(S,r) = \max\{\sum_{i\in T} x_i, W((S\setminus T)', r-|T|)\}$ so that $y > W(S,r)$ implies $y > \sum_{i \in T} x_i$ and $y > W((S \setminus T)', r-|T|)$.

Let $(i_a^\circ)_{a=1}^r$ be a sequence of members and, $\forall a \in \{1,\dots,r\}$, let $S_a^\circ = S \setminus \cup_{c=1}^{a-1}\{i_c^\circ\}$. Suppose that the sequence of members $(i_a^\circ)_{a=1}^r$ is such that $i_a^\circ \in T$ $\forall a \in \{1,\dots,|T|\}$, $i_{|T|+1}^\circ = \max S \setminus T$, and $i_a^\circ \in S_a^\circ$ $\forall a \in \{|T|+2,\dots,r\}$. Let $E = \{|T|+1,\dots,r\}$. Let $t_a^\circ = \delta^{r+1-a} x_{i_a^\circ}$ for any $i_a^\circ \in T$ and $t_a^\circ = \delta^{r+1-a} x_{i_a^\circ}(1-\delta)$ for any $i_a^\circ \in E$. Consider $\tilde{\sigma}'$ identical to $\tilde{\sigma}$ except that the leader, $\forall a \in \{1,\dots,r\}$, approaches member i_a° with an offer t_a° at any history that corresponds to state $(S_a^\circ, r+1-a)$. We now argue that, $\forall a \in \{1,\dots,r\}$, member i_a° offered t_a° in state $(S_a^\circ, r+1-a)$ accepts. To see this, for any $i_a^\circ \in T$ we have $t_a^\circ = \delta^{r+1-a} x_{i_a^\circ}$ and hence member i_a° offered t_a° in state $(S_a^\circ, r+1-a)$

accepts. To see that any $i^\circ_a \in E$ accepts t°_a in $(S^\circ_a, r+1-a)$, it suffices to argue that any $i^\circ_a \in E$ is dispensable in $(S^\circ_a, r+1-a)$. Member $i^\circ_{|T|+1} = \max S \setminus T$ is approached in $(S^\circ_{|T|+1}, r-|T|) = (S \setminus T, r-|T|)$ when, by $y > W((S \setminus T)', r-|T|)$, dispensable. Because $i^\circ_{|T|+1}$ is dispensable in $(S^\circ_{|T|+1}, r-|T|)$, we have $y > W(S \setminus \cup_{c=1}^{|T|+1}\{i^\circ_c\}, r+1-(|T|+1))$. By Lemma 4 part 6, we have, $\forall a \in \{|T|+2, \ldots, r\}$, $W(S \setminus \cup_{c=1}^{|T|+1}\{i^\circ_c\}, r+1-(|T|+1)) \geq W(S \setminus \cup_{c=1}^{a}\{i^\circ_c\}, r+1-a)$. Thus, $\forall a \in \{|T|+2, \ldots, r\}$, $y > W(S \setminus \cup_{c=1}^{a}\{i^\circ_c\}, r+1-a)$ and thus i°_a is dispensable in $(S^\circ_a, r+1-a)$. Because the leader approaches all members in the $(i^\circ_a)_{a=1}^r$ sequence with offers the members accept, her payoff from $\tilde{\sigma}'$ is $\delta^r(y - \sum_{i \in T} x_i - \sum_{i \in E} x_i(1-\delta))$. Because $\delta > \bar{\delta} \geq \bar{\delta}_a$ and $y > \sum_{i \in T} x_i$, we have $0 < y - \sum_{i \in T} x_i - n x_n(1-\delta) \leq y - \sum_{i \in T} x_i - \sum_{i \in E} x_i(1-\delta)$, which establishes the desired contradiction.

We now prove part 4. Suppose, towards a contradiction, that $y < W(S,r)$ and the policy passes in some $\tilde{\sigma} \in \Sigma$. Fix $\tilde{\sigma} \in \Sigma$ in which the policy passes. Suppose $r = |S|$. By part 2c and Lemma 5 part 2, all members in S are approached on the equilibrium path and each member is approached in a state in which he is indispensable. The leader's payoff from $\tilde{\sigma}$ is thus $\delta^r(y - \sum_{i \in S} x_i) < 0$, where the inequality follows from $y < W(S,r)$, because $W(S,r) = \sum_{i \in S} x_i$ when $r = |S|$. The leader thus has a profitable deviation to stop at the initial history of $\Gamma(S,r)$, a contradiction.

Suppose $r < |S|$ and let T be the set of members approached on the equilibrium path when indispensable. By parts 2a and 5, the leader's equilibrium payoff is at most $\delta^r(y - \sum_{i \in T} x_i)$. From $y \notin \mathcal{L}$, we have $y \neq \sum_{i \in T} x_i$ and from $\tilde{\sigma} \in \Sigma$, we have $y - \sum_{i \in T} x_i \geq 0$. Thus $y - \sum_{i \in T} x_i > 0$. Moreover, by part 2c, T solves (4) and hence $y > W((S \setminus T)', r-|T|)$. Thus $y > \max\{\sum_{i \in T} x_i, W((S \setminus T)', r-|T|)\} \geq W(S,r)$, which establishes the desired contradiction.

This concludes the proof of the induction step from k to $k+1$, for a given $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$. Repeating the same step for all states $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$ completes the specification of the l and c functions that define the space of strategies Σ. ■

6.4 Proof of Proposition 8

Let $\mathbf{t} = (t_1, \ldots, t_n)$, where $t_i \geq 0$ for each $i \in N = \{1, \ldots, n\}$, be a profile of transfers and let $\mathbf{a} = (a_1, \ldots, a_n)$ be the profile of members' actions, where for each $i \in N$, $a_i = 0$ indicates rejection and $a_i = 1$ indicates acceptance.

Consider the leader's decision in the second period, that is, in a subgame starting with $\mathbf{t}$ and $\mathbf{a}$. The leader's equilibrium actions are as follows. If $\sum_{i \in N} a_i \geq q$ and $\sum_{i \in N} a_i t_i < y$, then the leader initiates a vote and hence the policy passes. If $\sum_{i \in N} a_i \geq q$ and $\sum_{i \in N} a_i t_i > y$, then the leader stops and hence the policy does not pass. If $\sum_{i \in N} \leq q-1$ then the leader stops or initiates a vote when $\sum_{i \in N} t_i = 0$ and stops when $\sum_{i \in N} t_i > 0$ and hence the policy does not pass.

Consider the members' decision in the first period given $\mathbf{t}$. Let $\Gamma(\mathbf{t})$ be the subgame that starts at the history in which the leader's offer is $\mathbf{t}$. Assume the sequence in which members move in $\Gamma(\mathbf{t})$ is $(1, 2, \ldots, n)$. This assumption is without loss of generality because the argument below does not invoke that x_i is weakly increasing in i. Let $H_i(\mathbf{t})$ be the set of histories in $\Gamma(\mathbf{t})$ in which member $i \in N$ moves. For any history $h \in \cup_{i \in N} H_i(\mathbf{t})$

in which member $i \in N$ moves, let $\mathbf{a}(h) = (a_1(h), \ldots, a_{i-1}(h))$ be the profile of actions of the members moving before i and let $\#h = \sum_{j\in\{1,\ldots,i-1\}} a_j(h) + \sum_{j\in\{i+1,\ldots,n\}} \mathbb{I}(t_j \geq x_j)$ be the number of members who either move before i and accepted at history h or move after i and have been offered transfers weakly above their loss. We prove the following two lemmas.[17]

Lemma 10. *Let $\sigma(\mathbf{t})$ be an equilibrium of $\Gamma(\mathbf{t})$. Suppose $\sum_{i\in N} t_i < y$. Consider $h \in \cup_{i\in N} H_i(\mathbf{t})$ at which member i moves. Then, in $\sigma(\mathbf{t})$ starting from h, the policy*

1. *passes if either $\#h = q-1$ and $t_i \geq x_i$ or $\#h \geq q$,*
2. *does not pass if either $\#h = q-1$ and $t_i < x_i$ or $\#h \leq q-2$.*

Proof. Fix $\mathbf{t}$, equilibrium $\sigma(\mathbf{t})$ of $\Gamma(\mathbf{t})$ and suppose that $\sum_{i\in N} t_i < y$. We proceed by backward induction.

Consider $h_n \in H_n(\mathbf{t})$ at which member n moves. Irrespective of n's action, in $\sigma(\mathbf{t})$ starting from h_n, the policy does not pass if $\#h_n \leq q-2$ and the policy passes if $\#h_n \geq q$. If $\#h_n = q-1$, then we have $\sum_{j\in\{1,\ldots,n-1\}} a_j(h_n) = q-1$ and hence the policy passes if member n accepts and does not pass if n rejects. The payoff from the two actions is $-x_n + t_n$ and 0 respectively. Because $\sigma(\mathbf{t})$ constitutes an equilibrium, in $\sigma(\mathbf{t})$ starting from h_n, n accepts and the policy passes if $-x_n + t_n \geq 0$ and rejects and the policy does not pass if $-x_n + t_n < 0$.

Now suppose the lemma holds for each history $h \in \cup_{i\in\{k+1,\ldots,n\}} H_i(\mathbf{t})$, where $k \in \{1, \ldots, n-1\}$. We need to prove that the lemma holds for each history $h_k \in H_k(\mathbf{t})$. Consider $h_k \in H_k(\mathbf{t})$ at which member k moves. Note that (i) if k accepts and $x_{k+1} \geq t_{k+1}$, then the game proceeds to h_{k+1} with $\#h_{k+1} = \#h_k$, (ii) if k accepts and $x_{k+1} < t_{k+1}$, then the game proceeds to h_{k+1} with $\#h_{k+1} = \#h_k + 1$, (iii) if k rejects and $x_{k+1} \geq t_{k+1}$, then the game proceeds to h_{k+1} with $\#h_{k+1} = \#h_k - 1$, and (iv) if k rejects and $x_{k+1} < t_{k+1}$, then the game proceeds to h_{k+1} with $\#h_{k+1} = \#h_k$.

If $\#h_k \leq q-2$, then the game either proceeds to h_{k+1} with $\#h_{k+1} \leq q-2$, in which case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_{k+1} by the induction hypothesis, or proceeds to h_{k+1} with $\#h_{k+1} = q-1$, in which case we have $x_{k+1} < t_{k+1}$ and the policy does not pass in $\sigma(\mathbf{t})$ starting from h_{k+1} by the induction hypothesis. In either case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_k.

If $\#h_k \geq q$, then the game either proceeds to h_{k+1} with $\#h_{k+1} \geq q$, in which case the policy passes in $\sigma(\mathbf{t})$ starting from h_{k+1} by the induction hypothesis, or proceeds to h_{k+1} with $\#h_{k+1} = q-1$, in which case we have $x_{k+1} \geq t_{k+1}$ and the policy passes in $\sigma(\mathbf{t})$ starting from h_{k+1} by the induction hypothesis. In either case the policy passes in $\sigma(\mathbf{t})$ starting from h_k.

If $\#h_k = q-1$ and member k accepts, then the game either proceeds to h_{k+1} with $\#h_{k+1} = q$, in which case the policy passes in $\sigma(\mathbf{t})$ starting from h_{k+1} by the induction hypothesis, or proceeds to h_{k+1} with $\#h_{k+1} = q-1$, in which case we have $x_{k+1} \geq t_{k+1}$ and the policy passes in $\sigma(\mathbf{t})$ starting from h_{k+1} by the induction hypothesis. In either case the policy passes in $\sigma(\mathbf{t})$ starting from h_k when member k accepts.

If $\#h_k = q-1$ and member k rejects, then the game either proceeds to h_{k+1} with $\#h_{k+1} = q-2$, in which case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_{k+1} by the

[17] Whenever the leader offers transfers the payoffs are received in the second period. The lemmas thus work with the un-discounted payoffs in order to minimize the notation.

induction hypothesis, or proceeds to h_{k+1} with $\#h_{k+1} = q - 1$, in which case we have $x_{k+1} < t_{k+1}$ and the policy does not pass in $\sigma(\mathbf{t})$ starting from h_{k+1} by the induction hypothesis. In either case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_k when member k rejects.

The payoff of member k from the two actions at h_k with $\#h_k = q-1$ is thus $-x_k + t_k$ and 0 respectively. Because $\sigma(\mathbf{t})$ constitutes an equilibrium, in $\sigma(\mathbf{t})$ starting from h_k with $\#h_k = q - 1$, k accepts and the policy passes if $-x_k + t_k \geq 0$ and rejects and the policy does not pass if $-x_k + t_k < 0$. ■

Lemma 11. *Let $\sigma(\mathbf{t})$ be an equilibrium of $\Gamma(\mathbf{t})$ and let $\mathbf{a}$ be the equilibrium members' action profile. Suppose the policy passes in $\sigma(\mathbf{t})$. Then*

1. $\sum_{i\in N} a_i t_i = \sum_{i\in N} t_i$ *and*
2. $|\{i \in N | t_i \geq x_i\}| \geq q$.

Proof. Fix $\mathbf{t}$, equilibrium $\sigma(\mathbf{t})$ of $\Gamma(\mathbf{t})$, the equilibrium members' action profile $\mathbf{a}$ and suppose that the policy passes in $\sigma(\mathbf{t})$.

Part 1: It suffices to prove that, $\forall i \in N$, $t_i > 0$ implies $a_i = 1$. Suppose, towards a contradiction, that $t_i > 0$ and $a_i = 0$ for some $i \in N$. The payoff of member i from the equilibrium action $a_i = 0$ is $-x_i$ because the policy passes in $\sigma(\mathbf{t})$. Suppose member i deviates to $a'_i = 1$. The payoff from the deviation is either at least 0, when the policy does not pass following the deviation, or $-x_i + t_i$, when the policy passes following the deviation. In either case, the deviation to a'_i is profitable.

Part 2: It suffices to prove that for any $h \in \cup_{i\in N} H_i(\mathbf{t})$ at which member i moves, the policy does not pass in $\sigma(\mathbf{t})$ starting from h if either $\#h = q-1$ and $t_i < x_i$ or $\#h \leq q-2$. We proceed by backward induction.

Consider $h_n \in H_n(\mathbf{t})$ at which member n moves. If $\#h_n \leq q - 2$, then the policy does not pass in $\sigma(\mathbf{t})$ starting from h_n irrespective of member n's action. If $\#h_n \leq q - 1$, we have $\sum_{j\in\{1,\dots,n-1\}} a_j(h_n) = q - 1$ and hence the policy does not pass if member n rejects and either does not pass or passes if member n accepts. In the former case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_n. In the latter case, because the payoff from rejection is 0 while the payoff from acceptance is $-x_n + t_n$, because $t_n < x_n$, and because $\sigma(\mathbf{t})$ constitutes an equilibrium, member n rejects and the policy does not pass in $\sigma(\mathbf{t})$ starting from h_n.

Now suppose that for each history $h \in \cup_{i\in\{k+1,\dots,n\}} H_i(\mathbf{t})$ at which member i moves, where $k \in \{1, \dots, n-1\}$, the policy does not pass in $\sigma(\mathbf{t})$ starting from h if either $\#h = q-1$ and $t_i < x_i$ or $\#h \leq q - 2$. We need to prove that for each history $h_k \in H_k(\mathbf{t})$ at which member k moves, the policy does not pass in $\sigma(\mathbf{t})$ starting from h_k if either $\#h_k = q - 1$ and $t_k < x_k$ or $\#h_k \leq q - 2$. Consider $h_k \in H_k(\mathbf{t})$ at which member k moves.

If $\#h_k \leq q - 2$, then the game either proceeds to h_{k+1} with $\#h_{k+1} \leq q - 2$, in which case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_{k+1} by the induction hypothesis, or proceeds to h_{k+1} with $\#h_{k+1} = q - 1$, in which case we have $x_{k+1} < t_{k+1}$ and the policy does not pass in $\sigma(\mathbf{t})$ starting from h_{k+1} by the induction hypothesis. In either case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_k.

If $\#h_k = q - 1$ and member k rejects, then the game either proceeds to h_{k+1} with $\#h_{k+1} = q - 2$, in which case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_{k+1} by the induction hypothesis, or proceeds to h_{k+1} with $\#h_{k+1} = q - 1$, in which case we have

$x_{k+1} < t_{k+1}$ and the policy does not pass in $\sigma(\mathbf{t})$ starting from h_{k+1} by the induction hypothesis. In either case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_k when member k rejects.

If $\#h_k = q-1$ and member k accepts, then the game proceeds to h_{k+1} and the policy, in $\sigma(\mathbf{t})$ starting from h_{k+1}, either does not pass or passes. In the former case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_k. In the latter case, because the payoff from rejection is 0 while the payoff from acceptance is $-x_k + t_k$, because $t_k < x_k$, and because $\sigma(\mathbf{t})$ constitutes an equilibrium, member k rejects and the policy does not pass in $\sigma(\mathbf{t})$ starting from h_k. ■

We now prove Proposition 8. For the first part, let σ be an equilibrium of the simultaneous vote-buying game with transfer promises. Suppose, towards a contradiction, that the policy passes in σ. Let $\mathbf{t}$ be the equilibrium profile of transfers and let $\mathbf{a}$ be the equilibrium members' action profile. Because σ is an equilibrium, $\sigma(\mathbf{t})$ is an equilibrium of $\Gamma(\mathbf{t})$. Thus, by Lemma 11 parts 1 and 2, $\sum_{i\in N} a_i t_i = \sum_{i\in N} t_i \geq \sum_{i=1}^{q} x_i$. Because $\sum_{i=1}^{q} x_i > y$, the leaders equilibrium payoff $\delta(y - \sum_{i\in N} a_i t_i) < 0$, and hence choosing to stop in the first period is a profitable deviation for the leader.

For the second part, let σ be an equilibrium of the simultaneous vote-buying game with transfer promises. Suppose, towards a contradiction, that the policy does not pass in σ. Then the leader's equilibrium payoff is at most 0. Consider a deviation for the leader to $\mathbf{t'} = (x_1, \ldots, x_q, 0, \ldots, 0)$. Let $\sigma(\mathbf{t'})$ be an equilibrium of $\Gamma(\mathbf{t'})$, which exists by standard backward induction argument. Because $\sum_{i\in N} t'_i = \sum_{i=1}^{q} x_i < y$, and because the initial history h_1 of $\Gamma(\mathbf{t'})$ satisfies $\#h_1 = q-1$ and $t_1 \geq x_1$, Lemma 10 part 1 implies that the policy passes in $\sigma(\mathbf{t'})$. Hence, by Lemma 11 part 1, the leader's payoff from the deviation is $\delta(y - \sum_{i\in N} a_i t'_i) = \delta(y - \sum_{i\in N} t'_i) = \delta(y - \sum_{i=1}^{q} x_i) > 0$. This proves part (a).

To prove the remaining parts, let σ be an equilibrium of the simultaneous vote-buying game with transfer promises and let $\mathbf{t}$ be the equilibrium profile of transfers. By part (a), the policy passes in σ. Thus $\sigma(\mathbf{t})$ is an equilibrium of $\Gamma(\mathbf{t})$ and the policy passes in $\sigma(\mathbf{t})$. Hence $|\{i \in N | t_i \geq x_i\}| \geq q$ by Lemma 11 part 2. Moreover, $\sum_{i\in N} t_i = \sum_{i=1}^{q} x_i$. If not, then $\mathbf{t'}$ would be a profitable deviation for the leader. Thus $t_i \in \{0, x_i\}$ $\forall i \in N$ and $|\{i \in N | t_i \geq x_i\}| = q$. ■

6.5 Proof of Proposition 9

We first prove part (a) by contradiction. Assume $y - t > \sum_{j\in S^r} x_j$ and there exists an equilibrium in $\Gamma(S, r, t)$ in which the policy does not pass. The leader's payoff in this equilibrium is 0. Consider the strategy of offering r members in S each $x_i + \varepsilon$. Since each member would accept the offer, the policy passes and the leader's payoff is $y - t - \sum_{j\in S^r} x_j - r\varepsilon > 0$ for $\varepsilon > 0$ sufficiently low. Hence, the leader has a profitable deviation, a contradiction.

We next prove part (b) by induction. First consider $|S| = r$. The same argument as in the proof of Proposition 1 shows that the policy doe not pass in any equilibrium.

Next, suppose that part (b) holds for $|S| - r \leq k$ where $0 \leq k < |S|$. We prove that it also holds for $|S| - r = k+1$. Suppose, towards a contradiction, that there exists an equilibrium in $\Gamma(S, r, t)$ in which the policy passes. Suppose in this equilibrium, the leader approaches a set of members $\hat{S} \subseteq S$ in the first period of $\Gamma(S, r, t)$. Suppose member i' is the last one who makes the acceptance/rejection decision in $\hat{S}$. Given the induction hypothesis, when all preceding members in $\hat{S}$ have rejected the offers, if i' rejects the leader's offer, then the policy does not pass in any equilibrium in the resulting subgame $\Gamma(S \setminus \hat{S}, r, t)$ since $y - t < \sum_{j \in (S \setminus \hat{S})^r} x_j$. Hence, when all preceding members in $\hat{S}$ have rejected the offers, member i' accepts the offer t'_i if and only if $t'_i \geq x'_i$. A similar argument shows that this is true for any member $i \in \hat{S}$. Given that the policy passes in equilibrium in $\Gamma(S, r, t)$, the transfer t_i offered to $i \in \hat{S}$ satisfies $t_i \geq x_i$. Note that in the subgame that follows the acceptance of the members in $i \in \hat{S}$, $\Gamma(S \setminus \hat{S}, r - |\hat{S}|, t + \sum_{i \in \hat{S}} t_i)$, we have $y - t - \sum_{i \in \hat{S}} t_i \leq y - t - \sum_{i \in \hat{S}} x_i$. Since $y - t < \sum_{j \in S^r} x_j$, it follows that $y - t - \sum_{i \in \hat{S}} x_i < \sum_{j \in (S \setminus \hat{S})^{r - |\hat{S}|}} x_j$ and therefore $y - t - \sum_{i \in \hat{S}} t_i < \sum_{j \in (S \setminus \hat{S})^{r - |\hat{S}|}} x_j$. By the induction hypothesis, the policy does not pass in any equilibrium in $\Gamma(S \setminus \hat{S}, r - |\hat{S}|, t + \sum_{i \in \hat{S}} t_i)$, a contradiction. Hence, part (b) holds. ■

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