EXISTENCE AND UNIQUENESS OF RECURSIVE EQUILIBRIA WITH AGGREGATE AND IDIOSYNCRATIC RISK

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In this paper, I study the existence and uniqueness of recursive equilibria in production economies with aggregate risk. The economy features a continuum of agents who, in addition to aggregate risk, face idiosyncratic shocks and borrowing constraints. In particular, I establish existence for the Aiyagari-Bewley growth model à la Krusell and Smith (1998). In contrast to the existing literature, I do not rely on compactness to establish a fixed point. I instead exploit the monotonicity property of the equilibrium model and rely on arguments from convex analysis. Furthermore, this methodology gives rise to a uniqueness result for the Aiyagari-Bewley economy which is not restricted to a risk aversion parameter smaller equal one.

Keywords: Existence, Uniqueness, Dynamic stochastic general equilibrium, Incomplete markets, Heterogeneous agents, Aggregate uncertainty, Convergence.

JEL Classification: C61, C62, D51, D52, E21.

1. INTRODUCTION

The renewed interest in inequality in recent years has sparked a wealth of novel research based on economic models where heterogeneity across agents arises due to idiosyncratic risk. Such models go back to a dynamic stochastic general equilibrium model by Bewley (1977) where agents face idiosyncratic income shocks affecting their wealth which was extended by Aiyagari (1994) to include a production technology. Aggregate risk leading to business cycles was added by Krusell and Smith (1998). Despite the importance of these models in economics, many theoretical questions surrounding existence and uniqueness of a solution to models with both aggregate and idiosyncratic risk remain open. The challenge lies in handling the cross-sectional distribution of the agents' idiosyncratic variables, which becomes an infinite-dimensional element of the state space. In addition, this distribution changes stochastically over time depending on the realization of the aggregate

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shocks. The aggregate variables evolve, in turn, depending on how the cross-sectional distribution changes.

Existence of solutions to heterogeneous agent models, in particular, the Aiyagari-Bewley growth model with aggregate risk, has first been examined by Miao (2006) who builds the existence argument on a fixed point of the value function which directly depends on the cross-sectional distributions. Cao (2016) improves this argument by treating the case of zero aggregate capital more carefully. However, as pointed out in Cheridito and Sagredo (2016b), these two approaches are subject to a flaw in the theoretical argument, namely that weak convergence of measures does not imply convergence of moments. Cheridito and Sagredo (2016a) provide an alternative proof. They derive the existence of a sequential equilibrium in first proving existence of a value function which depends on a series of fixed aggregates and in a second step, proving existence of a fixed point in the aggregates consistent with the agent's optimal choices. All of these papers mostly focus on sequential equilibria and build on fixed-point theory relying on compactness. Miao (2006) extends his results to recursive equilibria depending on the current state of exogenous and endogenous variables, the distribution and the current value function, but this extension is also prone to the Cheridito and Sagredo (2016b) critique. A recent work which establishes existence of simple recursive equilibria is Brumm et al. (2017). They present an existence result for recursive equilibria of a very generic model. To keep the generality, they focus on bounded utility and finitely many agents. Hence, the existence of recursive equilibria for the model of Krusell and Smith (1998), which features unbounded utility and a continuum of agents, is still an open question.

This paper contributes to closing that gap by taking a different approach to the existing literature. I consider a generic equilibrium model with aggregate risk and a continuum of heterogeneous agents who maximize their CRRA utility when trading in securities. I make the restrictive assumption that the securities' returns are explicitly defined in terms of the cross-sectional distribution and the exogenous shocks. This assumption limits the class of models to production economies like the Aiyagari-Bewley model or questions in partial equilibrium. For this reason, this paper does not span the same general set of models as in Brumm et al. (2017). However, for the restricted set of models, I am able to establish uniqueness which is otherwise far-fetched in models combining aggregate and idiosyncratic risk. My methodology differs from the existing literature in two aspects.

First, a recursive equilibrium in a heterogeneous agent model with a continuum of agents is defined as a set of functions which depend on the cross-sectional distribution. I develop an alternative representation by showing that there is an equivalent set of equilibrium functions which have a random variable instead of a distribution as an argument. Sets of random variables are typically well behaved, especially the set of square-integrable random variables. The advantage of this approach fully enfolds when considering the Euler equation of the equilibrium problem. As I work with the random variable of security holdings instead of their distribution, I can substitute this random variable into the Euler equations of the individual agents. This transforms the continuum of individual Euler equations which are linked by the market-clearing condition into one Euler equation on random variables. It significantly simplifies the problem.

The second aspect in which I depart from the existing literature lies in the type of fixed point argument I use to establish existence. In contrast to the existing literature, which relies on the compactness of the state space requiring bounded utility functions, this paper exploits the monotonicity properties of the model and can thus handle unbounded utility functions. This approach is inspired by a series of papers by Rockafellar (1969, 1970, 1976a,b). Rather than using fixed-point theory on compact spaces, it relies on results from convex analysis and monotone operator theory. I show that the generalized Euler equation on random variables is a maximal monotone operator which implies that there exists a convex Lagrangian which has the generalized Euler equation as its first-order condition. n other words, there exists a social planner who solves the heterogeneous agent model by optimizing. Furthermore, there exists a root of the social planner's generalized Euler equation if one can find a candidate policy at which the generalized Euler equation has a negative value and another policy at which it has a positive value. Since this equilibrium problem can be solved using arguments from convex analysis, uniqueness of the solution can be examined in a straightforward manner. When using fixed-point theory relying on compactness instead, it is typically much more difficult to investigate the uniqueness of a solution. A nice side-effect of exploiting the monotonicity properties of the equilibrium model is that there exists a straight-forward iterative procedure which is guaranteed to converge to the equilibrium solution.

This paper is also related to the strand of literature on models with idiosyncratic risk but without aggregate risk. Existence for various model specification has been shown by Kuhn (2013) Acemoglu and Jensen (2015) and Açıkgöz (2018). In fact, Kuhn (2013) relies on monotonicity arguments as well to treat unbounded utility functions, whereas, Açıkgöz (2018) achieves this by exploiting continuity. Interestingly, results on uniqueness for these models have been established by Light (2018) in discrete time and Achdou et al. (2017) in continuous time. However, in both cases uniqueness is only shown for a risk aversion smaller equal one. In contrast to this, my uniqueness result for the Aiyagari-Bewley economy includes risk aversion parameters greater than one.

The paper proceeds as follows. I first introduce a generic model framework. Second, I characterize the recursive equilibrium by functions depending on random variables which results in a generalized Euler equation substituting the continuum of individual Euler equations. In Section 4, I establish the monotonicity properties which are necessary for the existence of equilibria. The following section applies this general framework to the Aiyagari-Bewley economy with aggregate risk. Lastly, I introduce the corresponding convergent iterative procedure which can be used to compute the equilibrium numerically. Appendix A contains all proofs.

2. A GENERIC MODEL

Consider a discrete-time infinite-horizon model with a continuum of agents of measure one. There are two kinds of exogenous shocks, an aggregate shock and an idiosyncratic shock. The aggregate shock characterizes the state of the economy with outcomes in $\mathcal{Z}^{ag} \subset \mathbb{R}$. It follows a first-order Markov process with transition probability $\mathbb{P}(.|z^{ag}) : \mathcal{Z}^{ag} \times \sigma(\mathcal{Z}^{ag}) \to [0,1]$ defined on the generating Borel-σ-algebra. The idiosyncratic shock with outcomes in $\mathcal{Z}^{id} \subset \mathbb{R}$ represents the agent-specific risk. It is a first-order Markov process which is i.i.d. across agents and whose transition probability at any point in time t is conditional on the aggregate shocks $\mathbb{P}(.|z^{id}_{t-1}, z^{ag}_{t-1}, z^{ag}_t) : \mathcal{Z}^{id} \times \sigma(\mathcal{Z}^{id} \times \mathcal{Z}^{ag} \times \mathcal{Z}^{ag}) \to [0,1]$. I denote the compound exogenous process $\left(z^{ag}_t, z^{id}_t\right)_{t\geq 0}$ by $(z_t)_{t\geq 0} \in \mathcal{Z}$ with $\mathcal{Z} = \mathcal{Z}^{ag} \times \mathcal{Z}^{id}$. The only requirement I impose on the exogenous stochastic processes is square integrability.

Assumption 1 (Square integrability) The aggregate and idiosyncratic exogenous processes $(z^{ag}_t)_{t\geq 0}$ and $(z^{id}_t)_{t\geq 0}$ are square integrable, i.e., $\mathbb{E}[(z^{ag}_t)^2] < \infty$ and $\mathbb{E}[(z^{id}_t)^2] < \infty$ at any time point $t \in \mathbb{N}$.

This specification of the aggregate and idiosyncratic shock is fairly flexible. It does include finite state Markov chains as well as continuous Markov processes in discrete time. Linear growth ensures square integrability in the latter case.

Example Examples for both exogenous processes include the following.

(i) Finite Markov chain: Define a finite state space $\mathcal{S} = \{s_1, \ldots, s_N\}$. Then, $z_t \in \mathcal{S}$ with the transition probabilities being given by $\pi_{ij} = \mathbb{P}(z_t = s_i | z_{t-1} = s_j)$ if z is an aggregate process or $\pi_{ij} = \mathbb{P}(z_t = s_i | z_{t-1} = s_j, z^{ag}_t)$ if z is idiosyncratic.

(ii) AR(1) process: Assume a normally distributed innovation $\eta \sim N(0,\sigma^2)$ and define $z_{t+1} = c + az_t + \eta$ with c constant and $a \in [0,1]$. The dependency of the idiosyncratic shock on the aggregate shock can be achieved by letting the mean and/or volatility of η vary depending on the current aggregate outcome.

Agents can invest in n one-period securities. An agent's share of security $j \in \{1,\dots,n\}$ is denoted by $(x_t^j)_{t\geq 0}$. The security $j \in \{1,\dots,n\}$ pays a rate of return $(r_t^j)_{t\geq 0}$ after one holding period.[1] Each agent chooses her share of the securities and consumption such that they satisfy certain constraints. First, individual consumption must be positive at all times $c_t > 0$, $t \geq 0$, and security holdings are subject to a borrowing constraints $x_t^j \geq \bar{x}^j$, $t \geq 0$, where $\bar{x}^j \leq 0$ for $j \in \{1,\dots,n\}$. Second, given the initial holdings $x_{-1}^j \geq \bar{x}^j$, each agent adheres to a budget constraint, which equates individual consumption and current security holdings to current endowment and the return on previous holdings

$$(1) \qquad \sum_{j=1}^{n} x_t^j + c_t = e\left(z_t\right) + \sum_{j=1}^{n} (1 + r_t^j) x_{t-1}^j \, \forall\, t \geq 0.$$

The endowment process e is given exogenously. The returns are aggregate endogenous variables. They are defined through the equilibrium condition which aggregates over the security holdings to equalize demand and supply. There are two possibilities how returns and the equilibrium condition can be connected. In production economies, the returns are explicitly set by an optimizing firm and, thus, depend on the firm's aggregate capital demand. In asset markets, returns are implicitly defined by a zero or unit-net supply equilibrium condition. I restrict the analysis of this paper to the former case.

Assumption 2 Suppose that the equilibrium returns $(r_t^j)_{t\geq 0}$ for security j are of the form

$$r_t^j = f^j\left(z_t, S_t\right),$$

where S_t is the exogenously given process describing the vector of the securities' aggregate supply. The function $f^j : \mathcal{Z} \times \mathbb{R} \to \mathbb{R}$ is once continuously differentiable in its second argument and bounded below by $f(z,S) > -1$ for any $(z,S) \in \mathcal{Z} \times \mathbb{R}$.

[1] Note that I choose to work solely with assets which pay a variable return for a fixed price rather than with assets which have a fixed return and can be bought for a variable price as these two types of assets are equivalent. To convert an asset with variable return into an asset with fixed return, we can simply substitute the asset holding at the transaction date with $x_t^j = y_t^j p_t^j$, where the price is given by $p_t^j = 1/\mathbb{E}_t[1 + r_{t+1}^j]$. This leads to a fixed payout y_t^j.

Remark This is a strong assumption which restricts the set of models to production economies or partial equilibria which define returns explicitly in terms of the aggregate security supply. It does not cover models with zero or unit-net supply conditions where returns are implicitly defined which requires solving an inverse problem. I leave the extension of the methodology presented herein to this latter case to future research.

Agents optimize their utility. I assume that all agents have a time-separable CRRA utility with a risk aversion coefficient $\gamma > 0$ or logarithmic utility when $\gamma = 1$. Then, given an agent's initial security holdings $x^j_{-1} \geq \bar{x}^j$, the individual optimization problem reads

$$
\begin{aligned}
(2) \qquad \max_{\left\{c_t, x^j_t\right\} \in \mathbb{R}^{n+1}} \quad & \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma}\right] \\
\text{s.t.} \quad & \sum_{j=1}^{n} x^j_t + c_t = e\left(z_t\right) + \sum_{j=1}^{n} (1 + r^j_t) x^j_{t-1} \, \forall \, t \geq 0 \\
& c_t > 0, \, x^j_t \geq \bar{x}^j \, \forall \, j \in \{1, \ldots, n\}, \, t \geq 0,
\end{aligned}
$$

where $\beta \in (0, 1)$ is the time preference parameter.

Let me now introduce the cross-sectional distribution of the model. I use the methodology of Fubini extension by Sun (2006) to ensure the validity of the law of large numbers when aggregating over the continuum of agents with measure one. In particular, denote the atomless measure space of agents by $(I, \mathcal{I}, \lambda)$ with $\lambda(I) = 1$ and the sample probability space at time t by $(\mathcal{Z}^{id}, \sigma(\mathcal{Z}^{id}), P^{id})$ with $P^{id} = \mathbb{P}(.|z_{t-1}, z^{ag}_t)$. Let f be a measurable function mapping the Fubini extension $(I \times \mathcal{Z}^{id}, \mathcal{I} \boxtimes \sigma(\mathcal{Z}^{id}), \lambda \boxtimes P^{id})$ into $\mathbb{R}$. If the random variables $f(i, .)$ are essentially pairwise independent, then $f(i, .)$ have a common distribution μ for λ-almost all $i \in I$. The same holds for the samples $f(., z^{id})$. When f represents individual security holdings, we have that $x^j_t = f^j(i, z^{id}_t)$ for agent i and, thus, x^j_t is distributed according to $\mu^j_t : \mathcal{Z}^{id} \times [\bar{x}^j, \infty) \to [0, 1]$. Hence, I denote the cross-sectional distribution of agent-specific variables at the beginning of period t by $\mu_t : \mathcal{Z}^{id} \times \prod_{j=1}^{n} [\bar{x}^j, \infty) \to [0, 1]$. Note that the aggregate shocks cause the cross-sectional distribution to vary over time, which is indicated by the time subscript of μ_t.

The equilibrium conditions of the model aggregate over the cross-sectional distribution to equate the securities' demand and supply. Let $\mathbf{E}$ denote a linear aggregation operator on the cross-sectional distribution $\mathbf{E} : \mathcal{P}(\mathcal{Z}^{id} \times \prod_{j=1}^{n} [\bar{x}^j, \infty)) \to \mathbb{R}^n$ which computes the vector of aggregate security holdings. Then, the equilibrium condition reads

$$\mathbf{E}[\mu_t] = S_t, \tag{3}$$

where S denotes the exogenous process of aggregate supply of the securities. The equilibrium condition implies that the equilibrium returns depend on the aggregate security holdings.

Before I define the equilibrium for this model, let me clarify the time line with Figure 1. Note that I specify the time line slightly differently from existing papers.

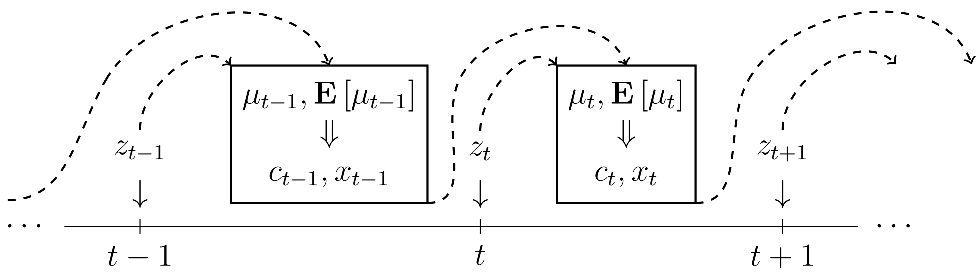

FIGURE 1. **Time line of events.** Before period t, the agent observes how much securities everybody decided to hold in the previous period. At period t, the agent observes the exogenous shocks z_t, and therefore, knows the beginning-of-period cross-sectional distribution μ_t and the aggregation quantity $\mathbf{E}[\mu_t]$. The agent then decides how much to consume c_t and how much to invest x_t^j in security j.

Often, x_t^j is substituted with x_{t+1}^j in the budget constraint (1) because this is the security holding with a payout at $t+1$. In contrast to that notation, however, I want to emphasize the time period, at which the agent optimally chooses the magnitude of her security holdings. Taking this view, the optimal consumption and security holdings choices have the same time subscript. My time line, therefore, indicates which filtration the endogenous variables are adapted to.

In a competitive equilibrium, the individual problems are solved subject to the equilibrium condition (3). In this thesis, I consider a particular competitive equilibrium of recursive form. To define a recursive equilibrium, I will switch to prime-notation for convenience, where a prime denotes variables in the current period and variables with no prime refer to the previous period.

Definition 3 (Recursive equilibrium) *Consider the measurable functions*[2]

$$
\begin{aligned}
g_c &: \mathcal{Z} \times \mathbb{R}^n \times \mathcal{P}\left(\mathcal{Z}^{id} \times \prod_{j=1}^{n}[\bar{x}^j, \infty)\right) \to \mathbb{R} \\
g_x &: \mathcal{Z} \times \mathbb{R}^n \times \mathcal{P}\left(\mathcal{Z}^{id} \times \prod_{j=1}^{n}[\bar{x}^j, \infty)\right) \to \mathbb{R}^n \\
g_r &: \mathcal{Z} \times \mathcal{P}\left(\mathcal{Z}^{id} \times \prod_{j=1}^{n}[\bar{x}^j, \infty)\right) \to \mathbb{R}^n, g(z,\mu) = f\left(z, \mathbf{E}[\mu]\right),
\end{aligned}
$$

where the equilibrium returns are given by Assumption (2) inserting the equilibrium condition (3). A solution to the agents' individual optimization problems (2) subject to the equilibrium condition (3) given an initial cross-sectional distribution of individual security holdings μ_0 is called recursive if for any point in time, the current equilibrium rates of return and the optimal consumption and security holdings choices for $j \in \{1, \ldots, n\}$ are given by

$$
\begin{aligned}
r^{j\prime} &= g_r^j\left(z', \mu'\right) \\
c' &= g_c\left(z', x^1, \ldots, x^n, \mu'\right) \\
x^{j\prime} &= g_x^j\left(z', x^1, \ldots, x^n, \mu'\right)
\end{aligned}
$$

for any agent with previous-period security holdings $(x^1, \ldots, x^n)$ who observes the current-period exogenous shock $z' = (z^{ag\prime}, z^{id\prime})$ and the beginning-of-current period cross-sectional distribution μ'.

3. CHARACTERIZING THE INCOMPLETE MARKETS EQUILIBRIUM

Now that the model and its equilibrium are defined in a general manner, I explain how the recursive equilibrium induces the law of motion of the cross-sectional distribution and how that leads to an operator which characterizes the equilibrium.

3.1. *A Consistent Law of Motion*

In a recursive equilibrium, we can easily define a law of motion of μ to μ' which is consistent with the agents' optimal choices. Given a fixed distribution μ over

[2] I assume that the equilibrium functions $g_.(.,.,\mu')$ are measurable w.r.t. the probability triple $\left(\mathcal{Z}^{id} \times \prod_{j=1}^{n}[\bar{x}^j, \infty), \sigma\left(\mathcal{Z}^{id} \times \prod_{j=1}^{n}[\bar{x}^j, \infty)\right), \mu'\right)$.

the cross-section of individual security holdings at the beginning of the previous period, the distribution changes in two steps $\mu \to \tilde{\mu} \to \mu'$. In the first step, the agents implement their optimal previous-period security holdings according to the recursive equilibrium from Definition 3, which leads to the end-of-previous period distribution

$$\tilde{\mu}\left(z^{id}, x\right) = \int_{\zeta \in \mathcal{Z}^{id} \cap \{\zeta \leq z^{id}\}} \int_{\bar{x}^1}^{\infty} \cdots \int_{\bar{x}^n}^{\infty} \mathbb{1}_{\{g_x(z^{ag}, \zeta, \chi, \mu) \leq x\}} d\mu\left(\zeta, \chi\right).$$

In the second step, the current-period shocks z' realize for all agents and shift the quantities of the agents with a specific idiosyncratic shock according to the exogenous transition probabilities conditional on the outcome of the aggregate shock. The beginning-of-current period distribution is hence, computed by integrating over the transition probabilities that the idiosyncratic state changes from z^{id} to $z^{id'}$ given the observed trajectory of z^{ag} to $z^{ag'}$. The distribution μ' is given by

$$\begin{aligned} \mu'\left(z^{id'}, x\right) &= \int_{\mathcal{Z}^{id}} \tilde{\mu}\left(z^{id}, x\right) \mathbb{P}\left(z^{id'} \middle| dz^{id}, z^{ag}, z^{ag'}\right) \\ &= \int_{z^{id} \in \mathcal{Z}^{id}} \int_{\zeta \in \mathcal{Z}^{id} \cap \{\zeta \leq z^{id}\}} \int_{\bar{x}^1}^{\infty} \cdots \int_{\bar{x}^n}^{\infty} \mathbb{1}_{\{g_x(z^{ag}, \zeta, \chi, \mu) \leq x\}} \\ &\quad d\mu\left(\zeta, \chi\right) \mathbb{P}\left(z^{id'} \middle| dz^{id}, z^{ag}, z^{ag'}\right). \end{aligned} \tag{4}$$

for all $z^{id'} \in \mathcal{Z}^{id}$ and $x \in \mathbb{R}^n$. Note that the rate of returns r' follow immediately from this definition of the current-period distribution according to Assumption (2).

3.2. *Rewriting the Recursive Equilibrium*

As we consider a heterogeneous agent model with a continuum of agents, the equilibrium defined in Definition 3 consists of policy functions which depend on the cross-sectional distribution. Since functions on measures are typically difficult to handle, I will show in this section that the equilibrium functions can be restated in a more tractable form. Inspired by the approach taken in a paper on mean field games by Carmona and Delarue (2015), I lift the functions on distributions into functions on the space of random variables. Carmona and Delarue (2015) trace the idea of converting distributions into random variables back to lectures on mean field games by Pierre-Louis Lions given at the Collège de France, a written account of which can be found in Cardaliaguet (2013). The reason why this conversion is possible is that the space of square-integrable random variables is equivalent to the Wasserstein space of probability measures. Therefore, one can establish mathematical properties like convergence in either space. However, taking derivatives is inherently simpler with respect to random variables. As I use Gâteaux derivatives to establish the existence result, I switch from distributions to random variables as suggested in the mean field game literature. Furthermore, I expect that most economists and econometricians are more familiar with the space of square-integrable random variables than with the more abstract Wasserstein space of probability measures.

To show that we can state the recursive equilibrium as a function of a random variable rather than the cross-sectional distribution, I consider an alternative way of defining the law of motion of the cross-sectional distribution. Instead of using c.d.f.s as in the previous section, I define it in terms of random variables. Since the initial distribution μ_0 is given, I use the corresponding probability space $(\mathcal{Z}^{id} \times \prod_{j=1}^{n}[\bar{x}^j,\infty), \sigma(\mathcal{Z}^{id} \times \prod_{j=1}^{n}[\bar{x}^j,\infty)), \mu_0)$ to define a conditional random variable representing the security holdings. Note that the conditional distribution of the security holdings is given by

$$\mu_0^{z^{id}}(x) = \frac{\mu_0\left(z^{id}, x\right)}{\mathbb{P}\left(z^{id} \middle| z_0^{ag}\right)}.$$

The conditional random variable of security holdings which is distributed according to $\mu_0^{z^{id}}$ is denoted by $\chi_0(z^{id})$. Given a trajectory of aggregate shocks $(z_t^{ag})_{t\geq 0}$, I define the series of conditional security holdings $(\chi_t(z^{id}))_{t\geq 0}$ by induction using (4)

$$\chi_t\left(z^{id'}\right) = \int_{\mathcal{Z}^{id}} g_x\left(z_{t-1}^{ag}, z^{id}, \chi_{t-1}\left(z^{id}\right), \mu_{t-1}\right) \mathbb{P}\left(z^{id'} \middle| dz^{id}, z_{t-1}^{ag}, z_t^{ag}\right), \ t \geq 1. \tag{5}$$

This implies that at any time t, the random variable of security holdings $\chi_t(z^{id})$ conditional on an idiosyncratic state z^{id} is a function of the trajectory of aggregate shocks $(z_0^{ag},\ldots,z_t^{ag})$ and the initial conditional random variable of security holdings $\chi_0(z^{id})$. Therefore, the security holdings at any time point are measurable w.r.t. μ_0, i.e., $\chi_t(z^{id}) \in L_{\mu_0}$ where I use the short-hand notation

$$L_{\mu_0} = L\left(\mathcal{Z}^{id} \times \prod_{j=1}^{n}[\bar{x}^j,\infty), \sigma\left(\mathcal{Z}^{id} \times \prod_{j=1}^{n}[\bar{x}^j,\infty)\right), \mu_0\right).$$

Accordingly, we can write any cross-sectional distribution μ_t as a function of $(z_0^{ag},\ldots,z_t^{ag})$ and $\chi_{t-1}(z^{id})$

$$\mu_t\left(z^{id}, x\right) = \int_{\zeta \in \mathcal{Z}^{id} \cap \{\zeta \leq z^{id}\}} \mu_0^{z_0^{id}}\left(\chi_t(\zeta) \leq x\right) \mathbb{P}\left(d\zeta \middle| z_t^{ag}\right), \tag{6}$$

which implies that the cross-sectional distribution at any time t is a measurable function w.r.t. the initial distribution $\mu_t \in L_{\mu_0}$. Due to (6), we can also rewrite the recursive equilibrium in terms of the conditional random variable of the beginning-of-current period security holdings $\chi\left(z^{id'}\right)$.

Proposition 4 (Recursive equilibrium with random variables) *Consider the recursive equilibrium in Definition 3. According to (6), we can rewrite the equilibrium functions with functions*

$$\begin{aligned} h_c :& \mathcal{Z} \times \mathbb{R}^n \times L_{\mu_{-1}} \to \mathbb{R} \\ h_x :& \mathcal{Z} \times \mathbb{R}^n \times L_{\mu_{-1}} \to \mathbb{R}^n \\ h_r :& \mathcal{Z} \times L_{\mu_{-1}} \to \mathbb{R}^n, g(z,\chi) = f\left(z, \mathbf{E}\left[\chi\right]\right), \end{aligned}$$

such that

$$\begin{aligned} r^{j'} &= g_r^j\left(z',\mu'\right) = h_r^j\left(z', \chi\left(z^{id'}\right)\right) \\ c' &= g_c\left(z', x^1, \dots, x^n, \mu'\right) = h_c\left(z', x^1, \dots, x^n, \chi\left(z^{id'}\right)\right) \\ x^{j'} &= g_x^j\left(z', x^1, \dots, x^n, \mu'\right) = h_x^j\left(z', x^1, \dots, x^n, \chi\left(z^{id'}\right)\right) \end{aligned}$$

where $\chi(z^{id'}) \in L_{\mu_0}$ is the conditional random variable of the beginning-of-current period security holdings of the form (5).

3.3. *The Euler Equation Operator*

In this section, I derive an operator which characterizes the equilibrium. Note that the optimal policy functions of the recursive equilibrium solve the Euler equation, which, if a suitable transversality condition holds, is necessary and sufficient for optimality. The set of Euler equations corresponding to the model from Section 2 reads

$$\begin{aligned} 0 = &\left(e\left(z'\right) + \sum_{j=1}^{n}\left[\left(1+r^{j'}\right)x^j - x^{j'}\right]\right)^{-\gamma} \\ &- \mathbb{E}^{(z''|z')}\left[\beta\left(1+r^{j''}\right)\left(e\left(z''\right) + \sum_{j=1}^{n}\left[\left(1+r^{j''}\right)x^{j'} - x^{j''}\right]\right)^{-\gamma}\right] \\ &+ y^{j'}, \forall j \in \{1,\dots,n\}. \end{aligned}$$

Note that I attach the borrowing constraints of the security holdings x' with Lagrange multipliers y'. Inserting the recursive equilibrium functions according to Definition 3, yields

$$\begin{aligned}
0 = &\left(e\left(z'\right) + \sum_{j=1}^{n} \left[\left(1 + g_r^j\left(z', \mu'\right)\right) x^j - g_x^j\left(z', x, \mu'\right) \right] \right)^{-\gamma} \\
&- \mathbb{E}^{(z''|z')} \Big[\beta \left(1 + g_r^j\left(z'', \mu''\right)\right) \Big(e\left(z''\right) \\
&+ \sum_{j=1}^{n} \left\{ \left(1 + g_r^j\left(z'', \mu''\right)\right) g_x^j\left(z', x, \mu'\right) - g_x^j\left(z'', g_x\left(z', x, \mu'\right), \mu''\right) \right\} \Big)^{-\gamma} \Big] \\
&+ y^j\left(z', x, \mu'\right), \forall j \in \{1, \ldots, n\},
\end{aligned}$$

where the Lagrange multipliers are functions which complement g_x such that $y \perp g_x \geq 0$.

The set of Euler equations has to hold at any exogenous state for any agent in the economy which means that it has to hold for a.e. z' and μ'-a.e. x. Recall from the previous section that the beginning-of-period security holdings x can be defined by a random variable $\chi(z^{id'}) \in L_{\mu_0}$ of the form (5). In terms of this random variable, the Euler equations have to hold pointwise. By inserting this random variable into the equilibrium functions instead of x and applying Proposition 4, the equilibrium functions are of the form $h_x(z', \chi(z^{id'}), \chi(z^{id'}))$. For notational convenience, I drop one of the two identical random variables in the following. Before defining the operator which characterizes the equilibrium, I make the following assumption on the equilibrium functions from Proposition 4.

Assumption 5 (Square integrability)

(*i*) The initial conditional random variable of security holdings distributed according to the initial conditional cross-sectional distribution $\chi(z^{id}) \sim \mu_0^{z^{id}}$ is square integrable w.r.t. μ_0.

(*ii*) The equilibrium functions h_c, h_x and h_r from Proposition 4 at a fixed $\chi \in L_{\mu_0}$ are square integrable w.r.t. their components (z', x). The corresponding probability distribution is the product of the distribution of the exogenous shocks with the conditional distribution of security holdings μ'. Due to (6), we can write μ' in terms of μ_0, and therefore, I assume $h_c, h_x, h_r \in L^2_{\mathbf{P}}$ with

$$L^2_{\mathbf{P}} = L^2 \left(\mathcal{Z} \times \prod_{j=1}^{n} [\bar{x}^j, \infty), \sigma \left(\mathcal{Z} \times \prod_{j=1}^{n} [\bar{x}^j, \infty) \right), \mathbb{P}(.|z)\mu_0^{z_0^{id}} \right),$$

i.e., $\mathbf{P}(z',x) = \mathbb{P}(z'|z)\mu_0^{z_0^{id}}(x)$.

(iii) The endowment function e and the function f defining the equilibrium rate of return are square integrable w.r.t. the exogenous shocks.

I now define the operator characterizing the equilibrium on the space of square-integrable functions $L^2_{\mathbf{P}}$.

Definition 6 (Euler equation operator) *Suppose that Assumptions 2 and 5 hold. Then, the Euler equation operator corresponding to the model from Section 2 is defined by* $\mathbf{T} : L^2_{\mathbf{P}} \longrightarrow L^2_{\mathbf{P}}, h \mapsto [\mathbf{T}^1[h], \ldots, \mathbf{T}^n[h]]$ *with*

$$\begin{aligned}
&\mathbf{T}^i[h_x]\left(z', \chi\left(z^{id'}\right)\right) = \\
&\left(e\left(z'\right) + \sum_{j=1}^{n}\left\{\left(1 + f^j\left(z', \mathbf{E}\left[\chi\left(z^{id'}\right)\right]\right)\right)\chi^j(z^{id'}) - h^j_x\left(z', \chi\left(z^{id'}\right)\right)\right\}\right)^{-\gamma} \\
&- \mathbb{E}^{(z''|z')}\Bigg[\beta\left(1 + f^i\left(z'', \mathbf{E}\left[\chi'\left(z^{id''}\right)\right]\right)\right)\Bigg(e\left(z''\right) + \sum_{j=1}^{n} \\
&\left\{\left(1 + f^j\left(z'', \mathbf{E}\left[\chi'\left(z^{id''}\right)\right]\right)\right)h^j_x\left(z', \chi\left(z^{id'}\right)\right) - h^j_x\left(z'', \chi'(z^{id''})\right)\right\}\Bigg)^{-\gamma}\Bigg],
\end{aligned}$$

where $i \in \{1, \ldots, n\}$ *and*

$$\chi'\left(z^{id''}\right) = \int_{\mathcal{Z}^{id}} h_x\left(z', \chi\left(z^{id'}\right)\right)\mathbb{P}\left(z^{id''}\middle| dz^{id'}, z^{ag'}, z^{ag''}\right)$$

defines the law of motion of the random variable of security holdings in line with (5).

Remark Note that the Euler equation operator directly incorporates the equilibrium conditions (3) due to Assumption 2. Furthermore, the Euler equation operator summarizes the Euler equations of all agents which is possible by switching to the random variables χ.

To summarize, we obtain a candidate equilibrium solution by finding functions $h_x, y \in L^2_{\mathbf{P}}$ which solve the following equation

$$\mathbf{T}[h_x]\left(z', \chi\left(z^{id'}\right)\right) + y\left(z', \chi\left(z^{id'}\right)\right) = 0,\ h_x \perp y \geq 0, \tag{7}$$

P-almost surely. Given such a solution, the original recursive equilibrium functions are recovered by

$$\begin{aligned} g_r^j\left(z', \mu'\right) &= h_r^j\left(z', x, \chi\left(z^{id'}\right)\right) \\ g_c\left(z', x, \mu'\right) &= h_c\left(z', x, \chi\left(z^{id'}\right)\right) \\ g_x^j\left(z', x, \mu'\right) &= h_x^j\left(z', x, \chi\left(z^{id'}\right)\right), \end{aligned}$$

where $h_{\cdot}$ is evaluated at the random variable $\chi(z^{id'}) \sim \mu'(z', x)/\mathbb{P}(z^{id'}|z^{ag'})$. If this candidate solution additionally satisfies a suitable transversality condition, it is indeed an equilibrium solution. I explain in the next section how to ensure that solving (7) leads to an equilibrium solution.

4. EXISTENCE OF AN EQUILIBRIUM SOLUTION

As is shown in Stokey et al. (1989), the extension of existence results with bounded utility functions to unbounded utility functions like the case of CRRA is typically done via constant returns to scale. However, due to the idiosyncratic shocks, there is a disjunction between individual security holdings and their rates of returns which aggregate over the individual holdings. Each agent in the continuum has zero weight and cannot influence aggregates. Therefore, it can happen that the individual security holdings grow substantially for an agent, but the rate of return does not change significantly to counteract this growth. In terms of Stokey et al. (1989) this model, thus, falls into the category of unbounded returns. To establish existence, I rely on arguments of monotonicity because compactness cannot be proven.

As I do not rely on a standard fixed-point theorem, let me first state the main mathematical result which I use to establish existence.

Corollary 7 (Rockafellar (1969)) *Let $\mathcal{C}$ be a Hilbert spaces over $\mathbb{R}$, and let $\mathbf{M} : \mathcal{C} \to \mathcal{C}^*$ be a maximal monotone operator.*[3] *Suppose that there exists a subset $B \subset \mathcal{C}$ such that $0 \in \operatorname{int}(\operatorname{conv}(\mathbf{M}(B)))$. Then, there exists a $c \in \mathcal{C}$ such that $0 \in \mathbf{M}(c)$.*

Remark This corollary essentially generalizes the result that there exists a root for a continuous real function $f : \mathbb{R} \to \mathbb{R}$ if there exist two points $a, b \in \mathbb{R}$ with $f(a) > 0$ and $f(b) < 0$ to higher-order spaces. Note that requiring continuity

[3] Monotonicity (see e.g., Phelps, 1997; Bauschke and Combettes, 2017): Let $\mathcal{E}$ be a Hilbert space. An operator $\mathbf{M} : \mathcal{E} \to \mathcal{E}$ is called a monotone operator if for any two elements of its graph $(e, f), (\tilde{e}, \tilde{f}) \in G(\mathbf{M}) = \{(e, f) \in \mathcal{E}^2 | f \in \mathbf{M}(e)\}$ it holds that $\langle e - \tilde{e}, f - \tilde{f}\rangle \geq 0$. It is, additionally, called maximal monotone if any $(\tilde{e}, \tilde{f}) \in \mathcal{E}^2$ with $\langle e - \tilde{e}, f - \tilde{f}\rangle \geq 0 \, \forall \, (e, f) \in G(\mathbf{M})$ is necessarily also an element of the graph $(\tilde{e}, \tilde{f}) \in G(\mathbf{M})$.

is not enough for mappings on multidimensional spaces.[4] Instead, the operator needs to be maximal monotone. If this property is satisfied, the corollary requires a subset B in the domain of the operator such that the interior of the convex hull of the subset's image contains zero. If one finds two elements c_- and c_+ such that the image $\mathbf{T}(c_-)$ is negative and the image $\mathbf{T}(c_+)$ is positive, it is possible in the general case to construct a set B such that the corollary holds.

The goal is to apply this corollary to the left-hand side of equation (7). Before I can do so, however, I have to establish that the operator

$$\mathbf{M}[h_x, y](z', \chi(z^{id'})) = \mathbf{T}[h_x](z', \chi(z^{id'})) + y(z', \chi(z^{id'})),\ h_x \perp y \geq 0, \tag{8}$$

is maximal monotone. I proceed in two steps. First, I consider the unconstrained case where by definition $y = 0$. From the maximal monotonicity of $\mathbf{T}$, I then derive the same property for $\mathbf{M}$ in the constrained case.

4.1. *Maximal Monotonicity in the Unconstrained Case*

Let me first restrict $\mathbf{T}$ to the set of functions $h_x \in L^2_{\mathbf{P}}$ which are continuous in the security holdings variable χ which is denoted by $C(L^2_{\mathbf{P}})$. I now define an admissible set $\mathcal{H}_\epsilon$ and show that the Euler equation operator $\mathbf{T} : \mathcal{H}_\epsilon \subset C(L^2_{\mathbf{P}}) \to C(L^2_{\mathbf{P}})$ is maximal monotone. The proofs can be found in Appendix A.

Proposition 8 (Admissible set) *Consider the model from Section 2 and suppose that Assumptions 2 and 5 hold. For an arbitrary $\epsilon > 0$, define the subspace $\mathcal{H}_\epsilon \subset C(L^2_{\mathbf{P}})$ as the set of random variables for which the following inequalities are satisfied $\mathbf{P}$-a.s. for any element $h \in \mathcal{H}_\epsilon$ and $\chi \in L^2_{\mathbf{P}}$*

(*i*) *Limited bond holdings:*

$$\sum_{j=1}^{n} h^j(z', \chi) \leq e(z') + \sum_{j=1}^{n} \left(1 + f^j(z', \mathbf{E}[\chi])\right) \chi^j - \epsilon$$

[4] A simple counterexample is $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x, y) = [\log(x + y), (x + y)^3]$.

(ii) Bounded Gâteaux derivative: [5]

$$\left\langle \sum_{j=1}^{n} \delta h^{j}\left(z', \chi; \tilde{\chi}\right), \tilde{\chi} \right\rangle \leq \left\langle \sum_{j=1}^{n} \left(1 + f^{j}\left(z', \mathbf{E}\left[\chi\right]\right)\right) \tilde{\chi}^{j}, \tilde{\chi} \right\rangle + \left\langle \frac{\partial}{\partial x} f^{j}\left(z', \mathbf{E}\left[\chi\right]\right) \mathbf{E}\left[\tilde{\chi}\right] \chi^{j}, \tilde{\chi} \right\rangle$$

for any $\tilde{\chi} \in L^2_{\mathbf{P}}$, where $\langle ., . \rangle$ denotes the inner product in $L^2_{\mathbf{P}}$.
Then, $\mathcal{H}_\epsilon$ is a Hilbert space.

Remark The admissible set includes all functions of current security holdings h_x, which are continuous and grow at most linearly in the previous security holdings, and have bounded slope in any direction. The first condition of the admissible set ensures that consumption is positive. Furthermore, due to the at most linear growth of $h \in \mathcal{H}_\epsilon$, square integrability w.r.t. μ_0 is preserved under the composition $h \circ h$.

Using the fact that monotonicity is equivalent to $\langle \delta \mathbf{T}[h; \tilde{h}], \tilde{h} \rangle \geq 0$ and that a continuous monotone operator is maximal monotone, I establish maximal monotonicity for the Euler equation operator in the following Lemma. However, I need to make another assumption.

Assumption 9 Aggregating over all securities, the change in the rate of return has a negative cross-sectional correlation with the change in security holdings

$$\sum_{j=1}^{n} \left\langle \mathbb{E}^{(z''|z')} \left[\delta f^{j}\left(z'', \mathbf{E}\left[h\right]; \tilde{h}\right) \right], \tilde{h}^{j} \right\rangle \leq 0$$

for any $h, \tilde{h} \in \mathcal{H}_\epsilon$.

Assumption 9 describes the typical general equilibrium effect that current prices of a security rise when the aggregate demand, i.e., the cross-sectional average of current security holdings, increases. This means that the return on the current choice of security holdings decreases. This effect arises naturally in models where the returns are implicitly given, e.g., through a unit-net supply equilibrium condition as can be seen in the following illustrative example.

[5] Gâteaux derivative (see e.g., Zeidler, 1986b): Let $\mathcal{E}$ be a Hilbert space. The directional derivative of an operator $\mathbf{M} : \mathcal{E} \to \mathcal{E}$ at a point $e \in \mathcal{E}$ in the direction $\tilde{e} \in \mathcal{E}$ is defined by $\delta \mathbf{M}(e; \tilde{e}) = d/dt \mathbf{M}(e + t\tilde{e})|_{t=0}$.

Example Consider an asset pricing model with two securities. I specify this example according to the generic framework in Section 2. Each security pays a dividend after one period of holding the asset given by an exogenous process depending solely on the aggregate shock. The dividend yield is denoted by $d_i' = d_i(z^{ag'})$, $i \in \{1, 2\}$ and the dividend payout is denominated in units of the consumption good. At any period of time the current asset value is given by the amount of shares held by an agent multiplied with the current asset price p_i', $i \in \{1, 2\}$. Hence, the budget constraint reads

$$c' + p_1'x_1' + p_2'x_2' = e\left(z'\right) + \left(d_1\left(z^{ag'}\right) + p_1'\right)x_1 + \left(d_2\left(z^{ag'}\right) + p_2'\right)x_2.$$

Note that the budget constraint in the individual optimization problem (2) is recovered by substituting $x_i = \frac{\tilde{x}_i}{p_i}$. The rate of return for the current security holdings choice is, hence, given by

$$r_i'' = \frac{d_i\left(z^{ag''}\right) + p_i''}{p_i'} - 1.$$

The current price p_i' is defined by the equilibrium condition

$$1 = \int_{\zeta \in \mathcal{Z}^{id}} \int_{\bar{x}_1}^{\infty} \int_{\bar{x}_2}^{\infty} x_i' d\mu'(\zeta, x_1', x_2')$$

normalizing the total amount of shares of security i to one. Since equilibrium prices are aggregate variables, they are given by recursive equilibrium functions $p_i' = g_{p_i}(z^{ag'}, \mu')$. Therefore, current prices depend on the current choice of security holdings which yields that the rate of return for the current security holdings choice r_i'', which is paid out in the next period, depends on the current price p_i'. If the price increases, the rate of return decreases. In equilibrium, prices i are positively related to the aggregate security holdings i which follows from the first-order condition of the individual optimization problem

$$\begin{aligned} p_1'x_1' + p_2'x_2' =& e\left(z'\right) + \left(d_1(z^{ag'}) + p_1'\right)x_1 + \left(d_2(z^{ag'}) + p_2'\right)x_2 \\ & - \left(\mathbf{E}^{(z''|z')}\left[\frac{d_i(z^{ag''}) + p_i''}{p_i'}\left(c''\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}. \end{aligned}$$

Aggregating across agents yields

$$p_i' = \frac{\int_{\mathcal{Z}^{id}} \int_{\bar{x}_1}^{\infty} \int_{\bar{x}_2}^{\infty} \left(\mathbf{E}^{(z''|z')}\left[\left(d_i(z^{ag''}) + p_i''\right)\left(c''\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} d\mu'(z^{id'}, x_1', x_2')}{\int_{\mathcal{Z}^{id}} e\left(z'\right) d\mathbb{P}\left(z^{id'} \,|\, z^{ag'}\right) + d_1(z^{ag'}) + d_2(z^{ag'})}.$$

The only variable on the right-hand side which depends on changes of x_i' is c''. The admissible set for the consumption function in equilibrium requires that the aggregated Gâteaux derivative of consumption w.r.t. changes in the cross-sectional random variable is positive (see Assumption (ii) in Proposition 8). Therefore, prices increase with higher security holdings in the aggregate or in other words, the cross-sectional correlation of prices and security holdings is positive.

Contrary to this illustrative example, however, I restrict the type of models in this paper to the case of explicitly defined returns by Assumption 2. Therefore, I have to add Assumption 9 to preserve the general equilibrium effect. In fact, Assumption 9 demands a slightly weaker general equilibrium effect than in the illustrative example as it aggregates over all assets, whereas, the example features a negative relation between returns and aggregate security holdings per asset. Under this assumption, maximal monotonicity holds.

LEMMA 10 (Maximal monotone Euler equation operator) *Suppose that Assumption 9 and the assumptions of Proposition 8 hold. Then, the Euler equation operator* $\mathbf{T} : \mathcal{H}_\epsilon \subset C(L^2_{\mathbf{P}}) \to C(L^2_{\mathbf{P}})$ *specified in Definition 6 is maximal monotone.*

Note that the operator can be extended to the whole domain $C(L^2_{\mathbf{P}})$ in a way which preserves maximal monotonicity (see e.g. Phelps, 1997). In our case, this means that one can define $\bar{\mathbf{T}}$ on $C(L^2_{\mathbf{P}})$ such that $\bar{\mathbf{T}}[h] = \mathbf{T}[h]$ for all $h \in \mathcal{H}_\epsilon$.

4.2. *Maximal Monotonicity in the Constrained Case*

To show that the operator $\mathbf{M}$ is maximal monotone also in the constrained case where the Lagrange multiplier is not necessarily zero, I introduce a Lagrangian which has the Euler equation operator as the sole element of its subdifferential. Once, we have such an objective function, we can add the borrowing constraints using the Lagrange multiplier y. The operator which ensures optimality of this newly introduced Lagrangian is then also maximal monotone.

It is important to understand that the first-order condition of the individual optimization problem (2) does not coincide with the Euler equation operator. The reason is that this optimization problem uses the sequential equilibrium policy, whereas, the Euler equation operator directly works with the recursive equilibrium functions and equilibrium returns. However, as $\mathbf{T}$ is maximal monotone, there exists a Lagrangian $L_{\mathbf{T}} : \mathcal{H}_\epsilon \times C(L^2_{\mathbf{P}}) \to [-\infty, \infty]$ such that $\mathbf{T}$ maximizes $L_{\mathbf{T}}$ in its second argument (see Ghoussoub, 2008, Theorem 5.1). According to (Lemma

5.1, Ghoussoub, 2008), the Lagrangian associated with $\mathbf{T}$ is given by

$$(9) \qquad L_{\mathbf{T}}(h,p) = \sup_{g \in \mathcal{H}_\epsilon} \{\langle p, g\rangle + \langle \mathbf{T}[g], h-g\rangle\}$$

and maximizing the Lagrangian over p for a given h yields $p^* = \mathbf{T}[h]$ with the function value $L_{\mathbf{T}}(h,p^*) = \langle p^*, h\rangle = \langle \mathbf{T}[h], h\rangle$. For notational convenience, I denote $L_{\mathbf{T}}(h,p^*)$ by $L_{\mathbf{T}}(h)$.

Remark The Lagrangian $L_{\mathbf{T}}$ aggregates over first-order Taylor approximations of the agents' utility over two time points. The aggregation denoted by the inner product happens w.r.t. the cross-sectional distribution. Therefore, we can interpret this Lagrangian as the objective function of a benevolent social planner. For each agent, the social planner uses a linearization of the agent's utility at two time points. As we are looking for a recursive equilibrium, summing the utility over two time points suffices to optimize in the infinite horizon. The social planner weights each agent equally since aggregation over the cross-sectional distribution evaluates each state by the amount of agents which currently have that same state.

Now that a suitable objective function associated with $\mathbf{T}$ is defined, I can attach the borrowing constraint $h \geq \bar{x}$. Therefore, I obtain a Lagrangian for the constrained problem $L : \mathcal{H}_\epsilon \times C(L^2_{\mathbf{P}}) \to [-\infty, \infty]$ given by

$$L(h,y) = L_{\mathbf{T}}(h) + \langle h - \bar{x}, y\rangle.$$

I show in the following the first-order conditions of this Lagrangian form a maximal monotone operator which implies the same for (8).

Lemma 11 (Maximal monotone $\mathbf{M}$) *Consider the model from Section 2 and suppose that Assumptions 2, 5 and 9 hold. Then, the operator associated with the constrained problem $\mathbf{M} : \mathcal{H}_\epsilon \subset C(L^2_{\mathbf{P}}) \to C(L^2_{\mathbf{P}})$ in (8) where $\mathcal{H}_\epsilon$ as in Proposition 8 is maximal monotone.*

4.3. *Sufficiency and Uniqueness*

Now that the property of maximal monotonicity is established for the left-hand side of the Euler equation (7), one can apply Corollary 7, which means that one has to find two points $(h_1, y_1), (h_2, y_2) \in \mathcal{H}_\epsilon \times C(L^2_{\mathbf{P}})$ such that $\mathbf{M}[h_1, y_1] > 0$ and $\mathbf{M}[h_1, y_1] < 0$ to obtain candidate solutions for the equilibrium problem. The Euler equation is normally only a necessary condition for the equilibrium. It needs to be verified that the candidate solution indeed maximizes individual utility.

Lemma 12 (Sufficiency) *Consider the model from Section 2 and suppose that Assumptions 2, 5 and 9 hold. Then, the Euler equation in (7) where the Euler equation operator is defined on $\mathcal{H}_\epsilon$ as in Proposition 8 is necessary and sufficient for an equilibrium solution.*

The sufficiency is mainly due to the fact that monotone operators are concepts from convex analysis. It is well known that the first-order condition is necessary and sufficient for a standard convex optimization problem. The problem, I consider in this paper is not standard but this property continues to hold.

Another property from standard convex analysis carries over which is uniqueness. A strictly convex optimization problem has a unique solution. I obtain an equivalent result due to strict monotonicity of the Euler equation operator.

Lemma 13 (Uniqueness) *Consider the model from Section 2 and suppose that Assumptions 2, 5 and 9 hold. Then, the Euler equation in (7) where the Euler equation operator is defined on $\mathcal{H}_\epsilon$ as in Proposition 8 has a unique solution.*

The uniqueness result refers to recursive equilibrium solutions in the set of continuous square-integrable functions. Even though there might exist other forms of sequential equilibria, I argue that recursive equilibria are the most important type of sequential equilibria for practical purposes. It is striking that the recursive equilibrium is unique for this fairly elaborate class of models with aggregate and idiosyncratic risk, especially given the wealth of literature on multiplicity of equilibria. It is well known that multiplicity can occur, for instance, in overlapping generations models, in the Arrow-Debreu setup or in bank run models. The main difference between these simpler setups and the one in this paper lies in the specification of risk and the type of equilibrium solution considered. In these simpler models, one typically solves for a steady-state equilibrium where large populations have to coordinate on finitely many possible actions. The coordination problem, i.e., the requirement to know which exact action the other agents choose, results in multiplicity. Morris and Shin (2000) show that this coordination problem is resolved and uniqueness obtained by introducing even a small amount of uncertainty about the other agent's behavior. The model investigated in this paper features the exact same remedy in form of idiosyncratic risk. A similar mechanism is at work in game theory when moving from pure strategy Nash equilibria to mixed strategy Nash equilibria, although the result is different as the uncertainty in mixed strategies ensures existence when pure strategies might fail.

Now, all the ingredients to establish existence are ready, the only step missing is to find two concrete feasible functions $(h_1, y_1), (h_2, y_2) \in \mathcal{H}_\epsilon \times C(L^2_{\mathbf{P}})$ such that the images of the Euler equation's left-hand side are negative and positive, respectively. To find specific points, we need more structure on the rate of returns. I will, therefore, illustrate the established results with the Aiyagari-Bewley growth model with aggregate risk. The strategy to identify these two points is also applicable to other models. One thing to keep in mind is that one has to choose $\epsilon > 0$ small enough in $\mathcal{H}_\epsilon$.

5. EXAMPLE OF THE AIYAGARI-BEWLEY GROWTH MODEL

I use the same growth model with aggregate shocks as in den Haan et al. (2010) and Krusell and Smith (1998). It is an Aiyagari-Bewley economy which fits the framework in this paper. The aggregate shock characterizes the state of the economy with outcomes in $\mathcal{Z}^{ag} = \{0, 1\}$ standing for a bad and good state, respectively. The idiosyncratic shock with outcomes in $\mathcal{Z}^{id} = \{0, 1\}$ indicates that an agent is unemployed or employed, respectively. Hence, the transition probabilities of the compound process $p^{z'|z}$ are exogenously given by a four-by-four matrix.

The security market consists of a claim to aggregate capital $(K_t)_{t\geq 0}$. An agent's share of physical capital is denoted by $(k_t)_{t\geq 0}$. The aggregation operator $\mathbf{E}[\mu_t] = K_t$ is hence defined by

$$K_t = \sum_{z^{id}=0}^{1} \int_{-\infty}^{\infty} k d\mu_t \left(z^{id}, k\right) \; \forall \, t \geq 0,$$

where μ_t is the cross-sectional distribution of idiosyncratic exogenous and endogenous variables at the beginning of time t, i.e. before the agents choose their optimal capital savings. Each agent chooses her share of physical capital and consumption such that they satisfy certain constraints. First, individual consumption must be positive at all times $c_t > 0$, $t \geq 0$, and capital holdings are subject to a hard borrowing constraint $k_t \geq \bar{x} = 0$, $t \geq 0$. Second, given an initial cross-sectional distribution μ_0[6] with non-negative support, each agent adheres to a budget constraint, which equates individual consumption and current capital stock to productive

[6] The initial cross-sectional distribution μ_0 does not only imply the initial aggregate capital K_0, but also the initial aggregate economic state as it is pinned down by the employment rate $\mathbb{P}(z_0^{id} = 1 | z_0^{ag}) = (1/K_0) \int_0^\infty k d\mu_0 \,(1, k)$.

income and saved capital stock[7]

$$k_t + c_t = I\left(z_t, k_{t-1}, K_t\right) + [1-\delta]\, k_{t-1}\, \forall\, t \geq 0, \tag{10}$$

where k_{t-1} is distributed according to $\mu_0/\mathbb{P}(z_0^{id}|z_0^{ag})$. The parameters in this budget constraint are defined as follows. The capital stock brought forward from period $t-1$ depreciates by a rate $\delta \in (0,1)$. The productive income is given by

$$\begin{aligned} I\left(z_t, k_{t-1}, K_t\right) &= R\left(z_t^{ag}, K_t\right) k_{t-1} \\ &\quad + z_t^{id}\pi\left[1-\tau_t\right] W\left(z_t^{ag}, K_t\right) + \left[1-z_t^{id}\right] \nu W\left(z_t^{ag}, K_t\right). \end{aligned} \tag{11}$$

It is composed of, first, the return on capital stock, and second, labor income, which equals the individual's wage W when the agent is employed and a proportional unemployment benefit νW otherwise. The agent's wage is subject to a tax rate $\tau_t = \nu(1-p_t^e)/(\pi p_t^e)$ whose sole purpose it is to redistribute money from the employed to the unemployed. The parameter $\nu \in (0,1)$ denotes the unemployment benefit rate, whereas, $p_t^e = \mathbb{P}(z_t^{id} = 1|z_t^{ag})$ is the employment rate at time t and $\pi > 0$ is a time endowment factor. It is reasonable to assume $\nu/\pi < 1 - \tau_t \Leftrightarrow \nu < \pi p_t^e$ for all $t \geq 0$. The wage W and the rental rate R are derived from a Cobb-Douglas production function for the consumption good

$$\begin{aligned} W\left(z_t^{ag}, K_t\right) &= (1-\alpha)\left(1 + z_t^{ag}a - (1-z_t^{ag})a\right)\left[\frac{K_t}{\pi p_t^e}\right]^{\alpha} \\ R\left(z_t^{ag}, K_t\right) &= \alpha\left(1 + z_t^{ag}a - (1-z_t^{ag})a\right)\left[\frac{K_t}{\pi p_t^e}\right]^{\alpha-1}, \end{aligned}$$

where $a \in (0,1)$ is the absolute aggregate productivity rate and $\alpha \in (0,1)$ is the output elasticity parameter. Labor supply is defined by the employment rate p_t^e scaled by the time endowment factor π.

The question is how this model fits the framework introduced earlier. There are two securities in this model, the share in physical capital $x_t^1 = k_t$ and the supplied labor $x_t^2 = 1$, which is not a choice variable as it is fixed by definition. The rates of return on these two components are given by

$$\begin{aligned} f^1\left(z_t, K_t\right) &= R\left(z_t^{ag}, K_t\right) - \delta \\ f^2\left(z_t, K_t\right) &= \left(z_t^{id}\pi\left[1-\tau_t\right] + \left[1-z_t^{id}\right]\nu\right) W\left(z_t^{ag}, K_t\right) - 1. \end{aligned} \tag{12}$$

[7]Note that I specify the time line slightly differently than den Haan et al. (2010) and Krusell and Smith (1998). These authors substitute k_t with k_{t+1} in the budget constraint (10) because this is the capital, which is put forward as start capital to period $t+1$. In contrast to that notation, however, I want to emphasize the time period, at which the agent optimally chooses the magnitude of her capital savings. Taking this view, the optimal consumption and capital savings choice have the same time subscript. My time line, therefore, indicates which filtration the endogenous variables are adapted to.

Therefore, Assumption 2 is satisfied. The rates of return are explicitly given. Assumption 9 that the returns are negatively related to changes in the security holdings is satisfied as well.

Proposition 14 *Assumption 9 is satisfied in the Aiyagari-Bewley economy with aggregate risk given in Krusell and Smith (1998) and den Haan et al. (2010).*

Assume that all agents have time-separable CRRA utility with a risk aversion coefficient $\gamma > 0$ or log-utility if $\gamma = 1$ and time preference parameter $\beta \in (0, 1)$. Then, given the initial cross-sectional distribution μ_0 with non-negative support, the individual optimization problem reads

$$(13) \qquad \begin{aligned} \max_{\{c_t, k_t\} \in \mathbb{R}^2} \; & \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma}\right] \\ \text{s.t.} \; & k_t + c_t = I\left(z_t, k_{t-1}, K_t\right) + [1-\delta]\, k_{t-1} \, \forall\, t \geq 0 \\ & c_t > 0,\, k_t \geq 0 \, \forall\, t \geq 0 \end{aligned}$$

where the productive income I is defined as in (11). I make the following technical assumption on the model parameters.

Assumption 15 Suppose that $\beta(1-\delta)^{1-\gamma} < 1$.

Remark Note that this assumption is trivially satisfied when $\gamma \leq 1$. Hence, this assumption is only necessary when $\gamma > 1$.

I can now apply the results from the previous section to establish existence and uniqueness of a solution. I show that the two points which result in the left-hand side of the Euler equation being greater and smaller than zero correspond to the save everything/consume nothing and the save nothing/consume everything strategies. From these two polar strategies, I construct a set which contains zero in its convex hull so that Corollary 7 can be applied.

Theorem 16 (Existence of a unique recursive equilibrium) *Consider the growth model together with Assumption 15. Define the admissible set $\mathcal{H}_\epsilon$ as in Proposition 8. Then, there exists a unique continuous square-integrable function $h \in \mathcal{H}_\epsilon$, $h : \mathcal{Z} \times \mathbb{R} \times L^2_{\mathbf{P}} \to \mathbb{R}$, which maximizes the individual optimization problem (13) with rates of returns as in (12).*

Remark It may seem surprising that I obtain uniqueness for the Aiyagari-Bewley economy with aggregate risk considering existing results in the literature on the Aiyagari-Bewley economy without aggregate risk. Light (2018) and Achdou et al. (2017) find uniqueness in discrete and continuous time, respectively, under the restriction that the risk aversion parameter $\gamma \leq 1$. Furthermore, Kuhn (2013) and Açıkgöz (2018) hint at potential multiplicity of equilibria for larger risk aversion. In contrast, I show uniqueness for a fairly general joint condition on the risk aversion parameter, the subjective discount factor and the depreciation rate, see Assumption 15. Risk aversion may be greater than one under this assumption. The question is how this can be reconciled with the multiplicity example given in Açıkgöz (2018). In that example, there are two equilibrating points. However, this example keeps the wage rate fixed and, thus, looks at the model from a partial equilibrium perspective disregarding the optimizing firm. Taking the firm with fixed wage rate into account, the equilibrium interest rate is uniquely determined by the model parameters. The question then is whether one of the equilibrating points of Açıkgöz (2018) coincides with the unique rate derived from the firm's first-order condition. It is not clear whether this necessarily has to be the case because the existence result in Açıkgöz (2018) is derived under the assumption that the depreciation rate $\delta \in (0, 1)$. However, in the numerical example $\delta = 1$. I conjecture that instead of multiplicity, we might actually obtain non-existence for this numerical example. Let me explain the intuition behind this conjecture. It is easy to verify that the parameter selection in the numerical example by Açıkgöz (2018) does not satisfy Assumption 15 which is due to $\delta = 1$. Theorem 16 on existence and uniqueness in this paper results from finding one strategy at which the Euler equation is larger than zero and one strategy at which it is less than zero. Assumption 15 ensures the former. If the assumption is not satisfied, both polar strategies save everything/consume nothing and save nothing/consume everything produce a negative value when inserted into the Euler equation. Due to the monotonicity of this equilibrium problem, non-existence rather than multiplicity of equilibria seems, therefore, more reasonable for the numerical example in Açıkgöz (2018).

6. AN ITERATIVE SOLUTION PROCEDURE

Due to the fact that I do not rely on compactness to establish existence for this type of model, the convergent iterative procedure of the contraction mapping theorem does not apply here. Hence, I cannot compute the equilibrium using value function iteration. However, the monotonicity approach leads to another convergent iterative procedure which is similar. This procedure is explained subsequently.

We can construct an iterative procedure P where $h^{n+1} = P(h^n)$ with h^n converging to a solution of (7) by exploiting the monotonicity of the Euler equation operator $\mathbf{T}$. To illustrate the idea, I will first look at the simplified problem without borrowing constraint. We can rewrite the Euler equation by

$$\mathbf{T}[h] = 0 \Leftrightarrow \mathbf{T}[h] + h = h \Leftrightarrow (\mathbf{T} + \mathbf{Id})[h] = h \Leftrightarrow h = (\mathbf{T} + \mathbf{Id})^{-1}[h],$$

where $\mathbf{Id}$ is the identity operator. The last equality contains the resolvent of the Euler equation $(\mathbf{T} + \mathbf{Id})^{-1}$. This operator has a very desirable property. It was shown by Minty (1962) that if the Euler equation operator is maximal monotone, its resolvent is firmly nonexpansive,[8] a property slightly stronger than Lipschitz continuity with coefficient one. It is well known that any firmly nonexpansive operator is equivalent to a mixture $(1/2)\mathbf{Id} + (1/2)\mathbf{R}$ of the identity operator $\mathbf{Id}$ and a nonexpansive operator $\mathbf{R}$ (see e.g., Bauschke and Combettes, 2017, Remark 4.34 (iii)). Weak convergence of the iteration of such a mixture to its fixed point is well established (see e.g., Zeidler, 1986a, Proposition 10.16). This procedure is also known as damped fixed-point iteration. Hence, iterating on the resolvent of a maximal monotone operator yields the proximal point algorithm. Therefore, iterating as in

$$h^{n+1} = (\mathbf{T} + \mathbf{Id})^{-1}[h^n],$$

where n is the iteration count, converges to the optimal policy, i.e. the root of the Euler equation operator.[9] This iterative procedure results in the proximal point algorithm. To understand how such a resolvent is constructed, let us look at a simplified example first.

Example (Resolvent of a subdifferential) Let $\mathcal{E}$ be a Hilbert space. Consider a lower semicontinuous proper convex function $F : \mathcal{E} \to [-\infty, \infty]$. It is well known that its subdifferential ∂F is maximal monotone (see e.g., Bauschke and Combettes,

[8] Nonexpansiveness (see e.g., Bauschke and Combettes, 2017): Let $\mathcal{E}$ be a Hilbert space. An operator $\mathbf{T} : \mathcal{E} \to \mathcal{E}$ is called nonexpansive if it is Lipschitz continuous with constant 1. It is called firmly nonexpansive if for all $e, \tilde{e} \in \mathcal{E}$ it holds that $\|\mathbf{T}(e) - \mathbf{T}(\tilde{e})\|^2 \leq \langle e - \tilde{e}, \mathbf{T}(e) - \mathbf{T}(\tilde{e})\rangle$.

[9] As a root, the optimal policy of the Euler equation operator represents an eigenfunction of the Euler equation operator's eigenvalue zero. This set of eigenfunctions is the same set which corresponds to the eigenvalue problem of the resolvent $\lambda\mathbf{Id} - (\mathbf{Id} + \mathbf{T})^{-1} = 0$ for the eigenvalue $\lambda = 1$. As the resolvent is Lipschitz continuous with coefficient one, which follows from the maximal monotonicity of the Euler equation operator, we can, in fact, characterize the resolvent's spectrum. The spectrum for nonlinear operators is not uniquely defined as the corresponding spectral theory is much more complex than for linear operators (see e.g. Appell et al., 2004). However, due to the Lipschitz property, we can use the definition by Kachurovskij leading to a compact spectrum with spectral radius of one. Hence, the optimal policy represents the eigenfunction corresponding to the resolvent's maximal eigenvalue.

2017, Theorem 20.48). We are looking for a fixed point $e^* \in \mathcal{E}$ of the resolvent of F, which can be computed by simple iteration with iteration count n,

$$e_n \overset{n\to\infty}{\longrightarrow} e^* \text{ with } e_{n+1} = (\partial F + \mathbf{Id})^{-1}(e_n).$$

The resolvent $(\partial F + \mathbf{Id})^{-1}$ can be represented by

$$\begin{aligned} e_{n+1} = (\partial F + \mathbf{Id})^{-1}(e_n) &\Leftrightarrow e_n = (\partial F + \mathbf{Id})(e_{n+1}) \\ &\Leftrightarrow 0 = (\partial F + \mathbf{Id})(e_{n+1}) - \mathbf{Id}(e_n) \\ &\Leftrightarrow e_{n+1} = \arg\min_{e\in\mathcal{E}} F(e) + \frac{1}{2}\|e - e_n\|^2. \end{aligned}$$

The latter is the update of the proximal point algorithm.[10]

This example shows that the proximal point algorithm in our case translates into an algorithm on augmented Lagrangians. To ensure convergence, a regularization term containing the previous iterate has to be added to the Lagrangian $L_{\mathbf{T}}$ in (9) associated with the left-hand side of the Euler equation. I follow Rockafellar (1976b) for defining the proximal point algorithm's update. The augmented Lagrangian is a function $L^A : \mathcal{H}_\epsilon \times L^2(\mathcal{Z}^{id} \times \mathbb{R}, \mathcal{B}(\mathcal{Z}^{id} \times \mathbb{R}), \mu) \to [-\infty, \infty]$ given by

$$\begin{aligned} L^A\left(h, y; z', \chi(z^{id'}), h^n\right) =& L_{\mathbf{T}}(h) \\ &+ \frac{1}{2\lambda}\|h - h^n\|^2_{L^2_{\mathbf{P}}} \\ &+ \begin{cases} -y(h-\bar{x}) + \frac{\lambda}{2}\|h-\bar{x}\|^2_{L^2_{\mathbf{P}}} & , h - \bar{x} \le \frac{y}{\lambda} \\ -\frac{1}{2\lambda}\|y\|^2_{L^2_{\mathbf{P}}} & , h - \bar{x} > \frac{y}{\lambda} \end{cases}, \end{aligned} \tag{14}$$

where $L_{\mathbf{T}}$ as in (9) and $\lambda > 0$ is the step size parameter of the proximal point algorithm. The first line of the augmented Lagrangian features the Lagrangian corresponding to the Euler equation operator from Definition 6. The second line consists of the objective's proximal point augmentation, which transforms the first-order condition into its resolvent. The last line corresponds to the inequality constraint. It also consists of the Lagrange term and the augmentation, but it is defined piecewise to account for the case of a binding constraint.

Remark The augmentation term in the Lagrangian (14) of the proximal point

[10] The proximal point update presented here is a simplified version. Rockafellar (1976a) proves convergence for the generalized resolvent $\lambda^n(\mathbf{Id} + 1/\lambda^n\mathbf{T})^{-1}$, also called Yosida approximation, where $\{\lambda^n\}_{n=1}^{\infty}$ is either constant and bounded away from zero or a series $0 < \lambda^n \nearrow \lambda^\infty \le \infty$.

algorithm does, in fact, represent a Tikhonov regularization. This regularization is necessary because the equilibrium correspondence is not a contraction mapping which implies that the proximal point algorithm can be interpreted as the equivalent to value function iteration for heterogeneous agent models. Furthermore, it has been shown in (Bauschke and Combettes, 2017, Theorem 27.23) that regularizations other than Tikhonov are admissible as well as long as the regularization function is uniformly convex in the policy h. An avenue for future research might, therefore, be to explore alternatives like the Sobolev regularization. However, one should keep in mind that the policies will not be differentiable everywhere when there are borrowing constraints.

With the augmented Lagrangian as above, I now state the algorithm to approximate a recursive equilibrium of the growth model in Algorithm 1.

Algorithm 1 Proximal point algorithm

▷ A Initialization

1: Set $n = 0$. Initialize the agents' choices individual capital and the Lagrange multiplier $H^n = (h^n, y^n)$.
2: Set the parameter $\lambda > 0$.
3: Set the termination criterion small $\tau > 0$ and the initial distance larger $d > \tau$.

▷ B Iterative procedure

4: **while** $d > \tau$ **do**
5: Update H^{n+1} by

$$h^{n+1}\left(z', \chi(z^{id'})\right) = \arg\min_{h \in \mathcal{H}_\epsilon} L^A\left(h, y^n; z', \chi(z^{id'}), h^n\right)$$
$$y^{n+1}\left(z', \chi(z^{id'})\right) = \max\left\{0, y^n\left(z', \chi(z^{id'})\right) - \lambda h^{n+1}\left(z', \chi(z^{id'})\right)\right\}$$

where L^A is defined as in (14).
6: Compute the distance $d = \|H^{n+1} - H^n\|_{L^2_{\mathbf{P}}}$.
7: Set $n = n + 1$.
8: **end while**

Remark Since the augmented Lagrangian L^A can be interpreted as the objective of a social planner optimizing the whole heterogeneous-agent economy, the proximal point algorithm is equivalent to the value function iteration of said social planner.

7. CONCLUSIONS

In this paper, I establish existence of recursive equilibria for production economies with a continuum of agents facing idiosyncratic shocks in combination with aggregate risk. Instead of relying on compactness arguments to establish the fixed point, I use the monotonicity of the equilibrium problem and arguments from convex analysis. Hence, I can also handle unbounded utility. An advantage of this approach is that it is easy to examine whether the equilibrium is unique. In particular, I establish existence and uniqueness of a recursive equilibrium for the Aiyagari-Bewley economy with aggregate risk not only for a risk aversion parameter of $\gamma \leq 1$ but also for economies with risk aversion $\gamma > 1$ satisfying the condition $\beta(1-\delta)^{1-\gamma} < 1$ where β is the subjective discount factor and δ denotes the depreciation rate.

APPENDIX A: PROOFS

A.1. *Proof of Proposition 8*

PROOF: It is well known that the subspace of continuous functions with bounded variation within L^2 is complete and hence, a Hilbert space itself. With condition (i), we take yet another subset of functions. It is easy to see that any limiting element h^* of a Cauchy sequence $h^n \in \mathcal{H}_\epsilon$, $n \in \{1, 2, \ldots\}$, satisfies conditions (i) as well. The subspace $\mathcal{H}_\epsilon$ is, therefore, complete and a Hilbert space itself. *Q.E.D.*

A.2. *Proof of Lemma 10*

PROOF: I compute the Gâteaux derivative of the Euler equation error to show monotonicity of $\mathbf{T}$. Let me first rewrite $\mathbf{T}$ in a simplified form

$$\begin{aligned} \mathbf{T}^i[h_x] = {} & h_c\left(z', \chi\left(z^{id'}\right)\right)^{-\gamma} \\ & - \mathbb{E}^{(z''|z')}\left[\beta\left(1 + f^i\left(z'', \mathbf{E}\left[\chi'\left(z^{id''}\right)\right]\right)\right) h_c\left(z'', \chi'(z^{id''})\right)^{-\gamma}\right], \end{aligned} \tag{15}$$

where $i \in \{1, \ldots, n\}$, h_c is given by the budget constraint and

$$\chi'\left(z^{id''}\right) = \int_{\mathcal{Z}^{id}} h_x\left(z', \chi\left(z^{id'}\right)\right) \mathbb{P}\left(z^{id''} \middle| dz^{id'}, z^{ag'}, z^{ag''}\right).$$

Its Gâteaux derivative is given by

$$
\begin{aligned}
\delta \mathbf{T}^i[h_x;\tilde{h}] = \gamma h_c\left(z', \chi\left(z^{id'}\right)\right)^{-\gamma-1} \sum_j^n \tilde{h}^j \\
+ \mathbb{E}^{(z''|z')}\Big[\beta\left(1 + f^i\left(z'', \mathbf{E}\left[\chi'\left(z^{id''}\right)\right]\right)\right) \\
\gamma h_c\left(z'', \chi'(z^{id''})\right)^{-\gamma-1} \delta h_c[h_x;\tilde{h}]\Big] \\
- \mathbb{E}^{(z''|z')}\left[\beta\delta\left(f^i\left(z'', \mathbf{E}\left[\chi'\left(z^{id''}\right)\right]\right);\tilde{h}\right) h_c\left(z'', \chi'(z^{id''})\right)^{-\gamma}\right]
\end{aligned}
$$

for $i \in \{1, \ldots, n\}$. The first two terms are non-negative $\mathbf{P}$-a.s. for any $h_x, \tilde{h} \in \mathcal{H}_\epsilon$ due to the conditions (i)-(ii) in Proposition 8. Since Assumption 9 holds, we obtain monotonicity as $\langle \delta\mathbf{T}[h_x;\tilde{h}], \tilde{h}\rangle \geq 0$. Furthermore, because $\mathbf{T}$ is continuous in h_x, we can apply (Corollary 20.28, Bauschke and Combettes, 2017) and obtain maximal monotonicity. *Q.E.D.*

A.3. *Proof of Lemma 11*

Before I start the proof, let me state some preliminaries. I need to show that the Lagrangian in (9) is a saddle function. Let me first define what a saddle function is in this context.

Definition 17 (Saddle function (see Rockafellar, 1970)) *(i) Let $\mathcal{C}$ and $\mathcal{D}$ be Hilbert spaces over $\mathbb{R}$. A saddle-function is an everywhere-defined function $L : \mathcal{C} \times \mathcal{D} \to [-\infty, \infty]$ such that $L(c, d)$ is a convex function of $c \in \mathcal{C}$ for any $d \in \mathcal{D}$ and a concave function of $d \in \mathcal{D}$ for any $c \in \mathcal{C}$.*

(ii) A saddle function is called proper if there exists a point $(c, d) \in \mathcal{C} \times \mathcal{D}$ with $L(c, \tilde{d}) < +\infty$ for any $\tilde{d} \in \mathcal{D}$ and $L(\tilde{c}, d) > -\infty$ for any $\tilde{c} \in \mathcal{C}$.

(iii) The operator associated with the saddle function L is defined as the set-valued mapping

$$
\begin{aligned}
\mathbf{T}_L(c, d) = \{(v, w) | L(\tilde{c}, d) - \langle \tilde{c}, v\rangle + \langle d, w\rangle \\
\geq L(c, d) - \langle c, v\rangle + \langle d, w\rangle \\
\geq L(c, \tilde{d}) - \langle c, v\rangle + \langle \tilde{d}, w\rangle \;\forall (\tilde{c}, \tilde{d}) \in \mathcal{C} \times \mathcal{D}\Big\},
\end{aligned}
$$

where $\langle ., .\rangle$ denotes the Hilbert space inner product. A saddle point is a point $(c^*, d^*) \in \mathcal{C} \times \mathcal{D}$ such that $0 \in \mathbf{T}_L(c^*, d^*)$.[11]

[11] The operator $\mathbf{T}_L$ is closely related to the subdifferential of the saddle function L as v equals the subgradient of $L(., d)$ at $c \in \mathcal{C}$ and w is the subgradient of $-L(c, .)$ at $d \in \mathcal{D}$.

Note that if our Lagrangian satisfies all properties of a saddle function, then the first-order conditions coincide with the operator $\mathbf{T}_L$. This operator can be further characterized by the following Corollary.

COROLLARY 18 (Rockafellar (1970)) *Let $\mathcal{C}$ and $\mathcal{D}$ be Hilbert spaces over $\mathbb{R}$. If $L(c,d)$ is a proper saddle function on $\mathcal{C} \times \mathcal{D}$, which is lower semicontinuous in its convex element $c \in \mathcal{C}$ and upper semicontinuous in its concave element $d \in \mathcal{D}$, then the operator $\mathbf{T}_L$ associated with L is maximal monotone.*

PROOF OF PROPOSITION 11: According to (Lemma 5.1 Ghoussoub, 2008), the Lagrangian $L_{\mathbf{T}}$ is convex and lower semicontinuous in $h \in \mathcal{H}_\epsilon$. It follows that the Lagrangian of the constrained problem L is convex and lower semicontinuous in $h \in \mathcal{H}_\epsilon$ and concave and upper semicontinuous in y. The Lagrangian is also a proper function because $L(h,y) > -\infty$ for any $h \in \mathcal{H}_\epsilon$ when $y = \mathbb{1}_{\{h<\bar{x}\}}$. Conversely, a security holding h can be constructed such that $L(h,y) < \infty$ for all $y \in C(L^2_{\mathbf{P}})$. We simply need $\mathbf{T}[h] < \infty$. This is achieved by setting consumption to a positive number $h_c = A < e$ which is less than the endowment e. Choosing the distribution χ such that the rate of returns are finite $f < \infty$, e.g., setting $\chi = \epsilon > 0$, leads to $h = \frac{1}{n}(e + \sum_{j=1}^{n}(1+f^j)\epsilon - A)$. Thus, Corollary 18 holds such that the first-order conditions summarized by $\mathbf{T}_L$ define a maximal monotone operator. The operator $\mathbf{M}$ is the subgradient of $L(.,y)$ at a Lagrange multiplier $y \in C(L^2_{\mathbb{P}})$ and, therefore, equals v in the definition of $\mathbf{T}_L$. Hence, maximal monotonicity of $\mathbf{M}$ follows trivially. *Q.E.D.*

A.4. *Proof of Lemma 12*

PROOF: Given that $h_x \in \mathcal{H}_\epsilon$ solves the Euler equation (7), we obtain a sequence of security holdings and consumption which by construction satisfy the first-order condition of the individual optimization problem and the equilibrium condition by

$$x^*_{t+1} = h_x\left(z_{t+1}, x^*_t, \chi_{t+1}\left(z^{id'}\right)\right)$$
$$c^*_{t+1} = e(z_{t+1}) + \sum_{j=1}^{n}\left\{\left(1 + f^j\left(z_{t+1}, \mathbf{E}\left[\chi_{t+1}\left(z^{id'}\right)\right]\right)\right)x^{j^*}_t - x^{j^*}_{t+1}\right\}$$

where $\chi_{t+1}(z^{id'})$ as in (5). Suppose that there is another arbitrary feasible series of security holdings and consumption in $L^2_{\mathbf{P}}$ such that $x_t \geq \bar{x}$, $t \geq 0$, and $c_t > 0$, $t \geq$

0, satisfying the budget constraint. Then, we get

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{t=0}^{T}\beta^t u(c_t^*) - u(c_t)\right] \\
\geq &\mathbb{E}\left[\sum_{t=1}^{T}\beta^t \left\langle \delta u(c_t^*; (x_{t-1}^* - x_{t-1})), (x_{t-1}^* - x_{t-1})\right\rangle\right] \\
&+ \mathbb{E}\left[\sum_{t=0}^{T}\beta^t \left\langle \delta u(c_t^*; (x_t^* - x_t)), (x_t^* - x_t)\right\rangle\right] \\
&= \mathbb{E}\left[\sum_{t=0}^{T-1}\beta^t \left\langle \delta\left(u(c_t^*) + \beta u(c_{t+1}^*); (x_t^* - x_t)\right), (x_t^* - x_t)\right\rangle\right] \\
&+ \beta^T \mathbb{E}\left[\langle \delta u(c_T^*; (x_T^* - x_T)), (x_T^* - x_T)\rangle\right],
\end{aligned}
$$

where the first term equals zero because the first-order condition holds **P**-a.e. For the same reason, the second term yields

$$
\begin{aligned}
&\beta^T \mathbb{E}\left[\langle \delta u(c_T^*; (x_T^* - x_T)), (x_T^* - x_T)\rangle\right] \\
&= -\beta^{T+1}\mathbb{E}\left[\left\langle \delta u(c_{T+1}^*; (x_T^* - x_T)), (x_T^* - x_T)\right\rangle\right] \\
&= +\beta^{T+1}\mathbb{E}\left[\left\langle (c_{T+1}^*)^{-\gamma}\left(x_T^* - x_T\right), (x_T^* - x_T)\right\rangle\right] \geq 0.
\end{aligned}
$$

As c_{T+1} is feasible, the marginal utility is positive. The Gâteaux derivative is negative due to the budget constraint. Hence, c_t^* is optimal. This concludes the proof. *Q.E.D.*

A.5. *Proof of Lemma 13*

PROOF: It is well known from convex analysis that the solution to a strictly convex optimization problem is unique (see e.g. Bauschke and Combettes, 2017). The Lagrangian in (9) is indeed strictly convex in h as $\langle \delta \mathbf{T}[h; \tilde{h}], \tilde{h}\rangle > 0$. This is obvious from the Gâteaux derivative in (15) as the first term is strictly positive due to the fact that consumption is bounded from above by $e(z') + \sum_{j=1}^{n}\{(1 + f^j(z', \mathbf{E}[\chi])) - \bar{x}^j\} < \infty$ for the constrained problem. *Q.E.D.*

A.6. *Proof of Proposition 14*

PROOF: To check that Assumption 9 holds, I show that

$$
\sum_{j=1}^{2}\left\langle \mathbb{E}^{(z''|z')}\left[\delta f^j\left(z'', \mathbf{E}\left[h\right]; \tilde{h}\right)\right], \tilde{h}^j\right\rangle \leq 0.
$$

Note that any direction $\tilde{h}$ has to equal zero in its second component which corresponds to the labor supply since this is not a choice variable. It is exogenously fixed at one. Therefore,

$$\left\langle \mathbb{E}^{(z''|z')}\left[\delta f^1\left(z'', \mathbf{E}\left[h\right]; \tilde{h}\right)\right], \tilde{h}^1\right\rangle \leq 0.$$

It is easy to see that the Gâteaux derivative of the return on capital at any z'' is negative

$$\begin{aligned} &\left\langle \delta f^1\left(z'', \mathbf{E}\left[h\right]; \tilde{h}\right), \tilde{h}^1\right\rangle \\ &= -(1-\alpha)\alpha\left(1 + z^{ag''}a - (1 - z^{ag''})a\right)\left(\pi p^{e''}\right)^{1-\alpha}(K'')^{\alpha-2}\,\mathbf{E}\left[\tilde{h^1}\right]^2 \leq 0 \end{aligned}$$

because the inner product computes the expectation with respect to individual capital which equals the aggregation over capital holdings denoted by the aggregation operator $\mathbf{E}$. This concludes the proof. *Q.E.D.*

A.7. *Proof of Theorem 16*

Proof of Theorem 16: I construct two savings policies h at which the left-hand side of the Euler equation is positive and negative, respectively. The idea is to use the two polar strategies save everything/consume nothing and save nothing/consume everything. From these two strategies, one can then in a last step construct a set of policy functions such that the convex hull of its image in the Euler equation operator contains zero.

Let me first define the candidate policy

$$h\left(z', k, \mu\right) = (1-\epsilon)\left(I\left(z', k, K\right) + (1-\delta)k\right),$$

where productive income I is as in (11). This implies that aggregate capital and current and next-period consumption are given by

$$\begin{aligned} K' &= (1-\epsilon)\left(\mathbb{E}^{\mu}\left[I\left(z', k, K\right)\right] + (1-\delta)K\right) \\ c\left(z', k, \mu\right) &= \epsilon\left(I\left(z', k, K\right) + (1-\delta)k\right) \\ c'\left(z', k, \mu\right) &= \epsilon\left[(1-\epsilon)^{\alpha} I\left(z'', h|_{\epsilon=0}, K'|_{\epsilon=0}\right) + (1-\epsilon)(1-\delta)h|_{\epsilon=0}\right]. \end{aligned}$$

Note that, due to the definition of c', its first variation equals $(1 - \delta + R\left(z'', K'\right))$.

Hence, the first-order condition is given by

$$\frac{\partial}{\partial c} u(c) - \beta \sum_{z'' \in \mathcal{Z}} p^{z''|z'} \left(1 - \delta + R(z'', K')\right) \frac{\partial}{\partial c} u(c').$$

Positive FOC: This outcome is equivalent to

$$1 > \beta \sum_{z'' \in \mathcal{Z}} p^{z''|z'} \left(1 - \delta + R(z'', K')\right) \left(\frac{c}{c'}\right)^{\gamma}, \tag{16}$$

where

$$\frac{c}{c'} = \frac{I(z', k, K) + (1-\delta)k}{(1-\epsilon)^{\alpha} I\left(z'', h|_{\epsilon=0}, K'|_{\epsilon=0}\right) + (1-\epsilon)(1-\delta)h|_{\epsilon=0}}.$$

I let $\epsilon > 0$ go to zero which is equivalent to the save everything/consume nothing strategy. Clearly, this strategy is admissible $h \in \mathcal{H}_{\epsilon^*}$ with $\epsilon^* = \min c(z', k, \mu) > 0$ getting closer and closer to zero. Also, it is easy to see that c/c' is an increasing function in individual capital in this case. I can compute its limit by applying l'Hôpital's rule

$$\lim_{k \to \infty} \frac{c}{c'} = \frac{1}{1 - \delta + R\left(z'', K'|_{\epsilon=0}\right)}.$$

If $\gamma = 1$, the right side of (16) equals $\beta < 1$ which results in the positive value of the first-order condition. When $\gamma \neq 1$, the right hand side is an increasing function of K'. It goes to zero when $K' \to 0$ and to $\beta(1-\delta)^{1-\gamma}$ when $K' \to \infty$. This also results in a positive value of the first-order condition by assumption. Hence, the trick is to choose $\epsilon^* > 0$ of the admissible set small enough.

Negative FOC: This outcome is equivalent to

$$1 < \beta \sum_{z'' \in \mathcal{Z}} p^{z''|z'} \left(1 - \delta + R(z'', K')\right) \left(\frac{c}{c'}\right)^{\gamma}.$$

I now let $\epsilon \to 1$ which corresponds to the save nothing/consume everything strategy. It is obvious that $c/c' \to +\infty$ in this case which gives us a negative value of the first-order condition. It remains to check that this strategy is admissible, i.e. $h \in \mathcal{H}_{\epsilon^*}$. Condition (i) of Proposition 8 is trivial, whereas, condition (ii) requires more care. Clearly, the left side of condition (ii) equals zero. Let me now analyze the first Gâteaux of income at $\kappa = (1-\epsilon)[I(z', k, K) + (1-\delta)k)]$ w.l.o.g. in the direction of $\tilde{\kappa}^{\epsilon} = (1-\epsilon)\tilde{\kappa}$

$$\begin{aligned}
\delta \mathbf{I}\left(z', \kappa; \tilde{\kappa}^{\epsilon}\right) &= \frac{\partial}{\partial K'} I(z', \kappa) \cdot \langle \tilde{\kappa}^{\epsilon}, \mathbf{1} \rangle + R(z', \kappa)\, \tilde{\kappa}^{\epsilon} \\
&= (1-\epsilon)^{\alpha}\, \delta \mathbf{I}\left(z', \kappa|_{\epsilon=0}; \tilde{\kappa}\right),
\end{aligned}$$

i.e., I can pull out ϵ from the derivative. It follows that the first derivative of income converges to zero for $\epsilon \to 1$. This shows that $h \in \mathcal{H}_{\epsilon^*}$.

Convex Hull: The last step consists of constructing a set of functions such that the convex hull of the image of that set contains zero. To do so, I define the savings policy by

$$h\left(z', k, \mu\right) = \left(1 - \epsilon\left(z', k, \mu\right)\right)\left(I\left(z', k, K\right) + (1-\delta)k\right),$$

where $\epsilon(z', k, \mu)$ is a continuous piecewise linear tent function which equals 1 everywhere except on $z'_{\text{fix}} \times (k^* - \Delta, k^* + \Delta) \times (K^* - \Delta, K^* + \Delta)$ and $\epsilon(z'_{\text{fix}}, k^*, K^*) = 0$. Now, define a grid of $k^*_n = n\Delta$ and $K^*_m = m\Delta$ such that we obtain a set of tent functions ϵ_{nm}. The question is which value the Euler equation operator has for such a tent function strategy. Similar to the analysis above, I compute the ratio of current-to-future consumption

$$\frac{c}{c'} = \frac{\epsilon_{nm} I\left(z', k, K\right) + (1-\delta)k}{\epsilon'_{nm}\left[(1-\epsilon_{nm})^{\alpha} I\left(z'', h|_{\epsilon=0}, K'|_{\epsilon=0}\right) + (1-\epsilon_{nm})(1-\delta)h|_{\epsilon=0}\right]},$$

where $\epsilon_{nm} = \epsilon_{nm}(z', k, K)$ and $\epsilon'_{nm} = \epsilon_{nm}(z'', h, K')$. Hence, whether $\mathbf{T}$ is positive or negative for a particular triplet (z', k, K) depends on the values of ϵ_{nm} and ϵ'_{nm}. Using a limit analysis as above, I distinguish four cases.

- $\epsilon_{nm} = 1$, $\epsilon'_{nm} \geq 0$: $\frac{c}{c'} \to \infty \Rightarrow \mathbf{T} < 0$
- $\epsilon_{nm} = 0$, $\epsilon'_{nm} \geq 0$: $\frac{c}{c'} = 0$ or $\lim_{k \to \infty} \frac{c}{c'} = \frac{1}{1-\delta+R(z'', K'|_{\epsilon=0})} \Rightarrow \mathbf{T} > 0$
- $0 < \epsilon_{nm} < 1$, $\epsilon'_{nm} = 0$: $\frac{c}{c'} \to \infty \Rightarrow \mathbf{T} < 0$
- $0 < \epsilon_{nm} < 1$, $\epsilon'_{nm} > 0$: $\mathbf{T}$ can be positive or negative

Hence, the Euler equation operator evaluated at the tent function strategy has a positive value at (z', k^*_n, K^*_m) and in the close vicinity of that point. It is negative elsewhere. Thus, the Euler equation operator evaluated at the tent function strategy is a tent function itself. This implies that one can find a convex combination of $\mathbf{T}_{nm}$ and the Euler equation operator $\mathbf{T} < 0$ at the save nothing/consume everything strategy which equals zero at (z', k^*_n, K^*_m). Therefore, the convex hull of the image of the set consisting of the two polar strategies and the set of tent function strategies with the multipliers ϵ_{nm} contains zero when the mesh size $\Delta \to 0$. Applying Corollary 7 ensures existence. Applying Lemma 13 yields uniqueness and concludes the proof. *Q.E.D.*

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